# Metric Spaces

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CHAPTER I

# Metric Spaces

ABSTRACT. The chapter introduces the initial definitions regarding metric spaces, subspaces, open and closed sets, and product spaces.

## 1. Metric Spaces

**Definition I.1.** Let $X$ be a set. A *metric* on $X$ is a function

$$\rho : X \times X \to \mathbb{R}$$

satisfying

**(M1)** $\rho(x, y) \geq 0$ and $\rho(x, y) = 0$ if and only if $x = y$ (Positivity);
**(M2)** $\rho(x, y) = \rho(y, x)$ (Symmetry);
**(M3)** $\rho(x, y) + \rho(y, z) \geq \rho(x, z)$ (Triangle Inequality).

The pair $(X, \rho)$ is called a *metric space*.

**Example I.2.** Let $X$ be any set and define $\rho : X \times X \to \mathbb{R}$ by

$$\rho(x, y) = \begin{cases} 0 & \text{if } x = y; \\ 1 & \text{otherwise }. \end{cases}$$

Then $\rho$ is a metric on $X$, called the *discrete metric*, and $(X, \rho)$ is called a *discrete metric space*.

*Proof.* In order to prove that $(X, \rho)$ is a metric space, we need to demonstrate **(M1)**, **(M2)**, and **(M3)**. We take the opportunity of this simple example to discuss some general assumptions we can make in the course of such a proof.

**(M1)** We have two things to verify; that the image of $\rho$ consists only of non-negative reals, and that $\rho(x, x) = 0$ for every $x \in X$. The first is immediate upon inspection of the definition, and in general won't need to be mentioned unless there is some doubt. The second is directly provided by the definition.

**(M2)** Let $x, y \in X$. If $x = y$, then $\rho(x, y) = 0 = \rho(y, x)$. Note that if **(M1)** if already verified, then this case need not be considered.

Thus suppose that $x$ and $y$ are distinct. Then $\rho(x, y) = 1 = \rho(y, x)$.

**(M3)** Let $x, y, z \in X$. If $x = z$, and **(M1)** is already verified, then this condition says that $0 \leq \rho(x, y) + \rho(y, z)$, which is true. If $x = y$ or $y = z$, this statement becomes an immediate equality. So again, we can assume that $x$, $y$, and $z$ are distinct. Then

$$\rho(x, z) = 1 < 2 = \rho(x, y) + \rho(y, z).$$

□

**Example I.3.** Let $x = \mathbb{R}$ and define $\rho(x, y) = |x - y|$. Then $(X, \rho)$ is a metric space.

*Proof.* We address **(M1)**, **(M2)**, and **(M3)**.

**(M1)** The absolute value is always nonnegative, and the absolute value of zero is zero, so $\rho(x, x) = |x - x| = |0| = 0$. On the other hand, if $x \neq y$, then $x - y \neq 0$, so $\rho(x, y) = |x - y| \neq 0$.

**(M2)** Let $x, y \in \mathbb{R}$; without loss of generality, assume that $x > y$. Then $\rho(x, y) = |x - y| = x - y = -(y - x) = |y - x| = \rho(y, x)$.

**(M3)** Let $x, y, z \in \mathbb{R}$; we have seen that $|a + c| \leq |a + b| + |b + c|$ for every $a, b, c \in \mathbb{R}$. Set $a = x - y$, $b = 0$, and $c = y - z$. Then $a + c = x - y$, $a + b = x - y$, and $b + c = y - z$. Thus

$$\rho(x, z) = |x - z| \leq |x - y| + |y - z| = \rho(x, y) + \rho(y, z).$$

□

**Example I.4.** Let $X = \mathbb{R}^k$ and define

$$\rho(x, y) = \sqrt{\sum_{i=1}^{k} (x_i - y_i)^2},$$

where $x = (x_1, \ldots, x_k)$ and $y = (y_1, \ldots, y_k)$.

*Remark.* The positivity and symmetry of $\rho$ are clear, but the proof of the triangle inequality is involved, and appears in the last section of this chapter, where it is generalized to the product of a finite number of metric spaces. □

**Example I.5.** Let $\mathbb{R}^\infty$ denote the set of all sequences of real numbers that are eventually zero, that is, sequences $(x_n)$ such that $x_n = 0$ for all but finitely many $n$. Let $X = \mathbb{R}^\infty$ and for $x, y \in X$, define

$$\rho(x, y) = \sqrt{\sum_{i=1}^{\infty} (x_i - y_i)^2},$$

where $x = (x_n)$ and $y = (y_n)$. This make sense, since there are only finitely many nonzero summands. Then $(X, \rho)$ is a metric space.

**Example I.6.** Let $\ell^2$ denote the set of all sequences of real numbers $(x_n)$ that satisfy the converge criterion

$$\sum_{i=1}^{\infty} x_i^2 < 0.$$

Let $X = \ell^2$ and for $x, y \in X$, define

$$\rho(x, y) = \sqrt{\sum_{i=1}^{\infty} (x_i - y_i)^2},$$

where $x = (x_n)$ and $y = (y_n)$. That this series converges follows from the inequality

$$(a \pm b)^2 \leq 2(a^2 + b^2),$$

which the reader is welcome to verify. Then $(X, \rho)$ is a metric space.

**Example I.7.** Let $X = \mathbb{Q}$ and let $p$ be a positive prime integer. For each $x \in \mathbb{Q}$, there exists unique $m, n, \alpha \in \mathbb{Z}$ such that $x = p^\alpha \frac{m}{n}$, where $\gcd(m, n) = 1$ and $p$ does not divide $m$ or $n$. The *p-adic norm* of $x$ is $|x|_p = \frac{1}{p^\alpha}$. Set $\rho(x, y) = |x - y|_p$. Then $(X, \rho)$ is a metric, known as the *p-adic metric* on $\mathbb{Q}$. Here one can show that not only does $\rho$ satisfy the triangle inequality, but also the stronger inequality $|x - y|_p \leq \max\{|x|_p, |y|_p\}$.

**Exercise I.1.** Let $\mathcal{F}_{[a,b]}$ denote the set of all bounded functions $f : [a, b] \to \mathbb{R}$. Let $X = \mathcal{F}_{[a,b]}$ and for $f, g \in X$ define

$$\rho(f, g) = \max\{|f(x) - g(x)| \mid x \in [a, b]\}.$$

Show that $(X, \rho)$ is a metric space.

**Exercise I.2.** Let $\mathcal{C}_{[a,b]}$ denote the set of all continuous functions $f : [a, b] \to \mathbb{R}$. Let $X = \mathcal{C}_{[a,b]}$ and for $f, g \in X$ define

$$\rho(f, g) = \int_a^b |f - g| \, dx.$$

Show that $(X, \rho)$ is a metric space.

## 2. Metric Subspaces

**Definition I.8.** Let $(X, \rho)$ be a metric space and let $A \subset X$. ' Let $\rho_A : A \times A \to \mathbb{R}$ be the restriction of $\rho$ to $A \times A \subset X \times X$. Then $\rho_A$ is a metric on $A$, and $(A, \rho_A)$ is called a *subspace* of $(X, \rho)$.

**Example I.9.** Let $X = \mathbb{R}^2$, and define

$$\rho : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R} \quad \text{by} \quad \rho(p_1, p_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2},$$

where $p_i = (x_i, y_i)$. Then $\rho$ is the standard metric on $\mathbb{R}^2$.

Define

$$\mathbb{S}^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}.$$

We call $\mathbb{S}^1$ the *unit circle*. It inherits the metric $\rho_{\mathbb{S}^1}$ from $(\mathbb{R}^2, \rho)$.

Define

$$\mathbb{D}^2 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}.$$

We call $\mathbb{D}^2$ the *(closed) unit disk*, and $(\mathbb{D}^2, \rho_{\mathbb{D}^2})$ is a metric space.

**Example I.10.** Let $\mathbb{S}^1$ be the unit circle, and let $\rho$ be as in Example I.9. We may define a metric

$$\alpha : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{R} \quad \text{by} \quad \alpha(p_1, p_2) = 2 \arcsin(\rho(p_1, p_2)).$$

where $p_1, p_2 \in \mathbb{S}^1$. Then $\alpha(p_1, p_2)$ is the angle, measured in radians, from $p_1$ to the origin and then to $p_2$; this is the arclength of the shortest path between these two points.

This produces a different metric on $\mathbb{S}^1$. In due course, we will investigate the relationship between these metrics and related consequences for the structure of the metric space.

**Example I.11.** Let $X = \mathbb{R}^3$, and define

$$\rho : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R} \quad \text{by} \quad \rho(p_1, p_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2},$$

where $p_i = (x_i, y_i, z_i)$. Then $\rho$ is the standard metric on $\mathbb{R}^3$.

Define

$$\mathbb{S}^2 = \{(x, y, z) \in \mathbb{R}^2 \mid x^2 + y^2 + z^2 = 1\}.$$

We call $\mathbb{S}^2$ the *unit sphere*, and $(\mathbb{S}^2, \rho_{\mathbb{S}^2})$ is a metric space.

Define

$$\mathbb{D}^3 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leq 1\}.$$

We call $\mathbb{D}^2$ the *(closed) unit ball*, and $(\mathbb{D}^3, \rho_{\mathbb{D}^3})$ is a metric space.

## 3. Product Metric Spaces

The definition of distance in $\mathbb{R}^k$ has been computed using $k-1$ applications of the Pythagorean Theorem; it is clearly the definition that we want. However, in order to apply results to $\mathbb{R}^k$ that we have proven from the metric axioms, we need to first prove that $\mathbb{R}^k$ is indeed a metric space. This involves a demonstration of the triangle inequality; that is, given

$$x = (x_1, \ldots, x_k), y = (y_1, \ldots, y_k), z = (z_1, \ldots, z_k),$$

we need to show that

$$\sqrt{\sum_{j=1}^{k}(x_j - z_j)^2} \leq \sqrt{\sum_{j=1}^{k}(x_j - y_j)^2} + \sqrt{\sum_{j=1}^{k}(y_j - z_j)^2}.$$

Proving this directly would make use of the triangle inequality

$$|a - c| \leq |a - b| + |b - c|$$

in $\mathbb{R}$, and an application of the Cauchy-Schwartz Inequality (below). With approximately the same effort, we can generalize this result to construct the product of a finite number of arbitrary metric spaces. The definition of distance in the product space is motivated by our previous use of the Pythagorean Theorem.

**Theorem I.12.** *Let $(X_1, \rho_1), \ldots, (X_n, \rho_n)$ be a finite collection of metric spaces. Let $X = \times_{k=1}^{n} X_k$, and define $\rho : X \times X \to \mathbb{R}$ by*

$$\rho(x, y) = \sqrt{\sum_{k=1}^{n} \rho_k(x_k, y_k)^2},$$

*where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$, and $x_k, y_k \in X_k$ for $k = 1, \ldots, n$. Then $(X, \rho)$ is a metric space.*

We call $\rho$ the *product metric* on $X$. The difficulty of the proof of this proposition lies in the triangle inequality, a computation which we will break into several intermediate results.

**Lemma I.13.** *Let $a_k, b_k \in \mathbb{R}$ for $k = 1, \ldots, n$. Then*

$$\sum_i \sum_j (a_i b_j - a_j b_i)^2 = 2 \sum_{i \neq j} (a_i^2 b_j^2 - a_i a_j b_i b_j).$$

*Proof.* Note that

$$(a_i b_j - a_j b_i)^2 = a_i^2 b_j^2 + a_j^2 b_i^2 - 2 a_i a_j b_i b_j.$$

Then

$$\begin{aligned}\sum_i \sum_j (a_i b_j - a_j b_i)^2 &= \sum_i \sum_j (a_i^2 b_j^2 + a_j^2 b_i^2 - 2 a_i a_j b_i b_j) \\ &= 2 \sum_{i \neq j} (a_i^2 b_j^2 - a_i a_j b_i b_j).\end{aligned}$$

□

**Lemma I.14.** *Let $a_k, b_k \in \mathbb{R}$ for $k = 1, \ldots, n$. Then*

$$\left(\sum_{k=1}^{n} a_k b_k\right)^2 = \sum_{k=1}^{n} a_k^2 \sum_{k=1}^{n} b_k^2 - \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(a_i b_j - a_j b_i)^2.$$

*Proof.* Compute that

$$\begin{aligned}\sum_k a_k^2 \sum_k b_k^2 &= \sum_k a_k^2 b_k^2 + 2\sum_{i\neq j} a_i^2 b_j^2 \\ &= \left(\sum_k a_k^2 b_k^2 + 2\sum_{i\neq j} a_i a_j b_i b_j\right) - 2\sum_{i\neq j} a_i a_j b_i b_j + 2\sum_{i\neq j} a_i^2 b_j^2 \\ &= \left(\sum_k a_k b_k\right)^2 + 2\left(\sum_{i\neq j} a_i^2 b_j^2 - \sum_{i\neq j} a_i a_j b_i b_j\right).\end{aligned}$$

Subtracting the equation of Lemma I.13 to both sides implies the result. □

**Lemma I.15** (Cauchy-Schwartz Inequality)**.** *Let $a_k, b_k \in \mathbb{R}$ for $k = 1, \ldots, n$. Then*

$$\left(\sum_{k=1}^{n} a_k b_k\right)^2 \leq \sum_{k=1}^{n} a_k^2 \sum_{k=1}^{n} b_k^2.$$

*Proof.* This follows from Lemma I.14 by noting that $\sum_{i=1}^{n}\sum_{j=1}^{n}(a_i b_j - a_j b_i)^2$ is always nonnegative. □

**Lemma I.16.** *Let $a_k, b_k, c_k \in \mathbb{R}$ be positive for $k = 1, \ldots, n$. Then*

$$\sqrt{\sum_{k=1}^{n} a_k^2} \leq \sqrt{\sum_{k=1}^{n} b_k^2} + \sqrt{\sum_{k=1}^{n} c_k^2}.$$

*Proof.* For $k = 1, \ldots, n$, we have $a_k \leq b_k + c_k$, so $a_k^2 \leq b_k^2 + c_k^2 + 2b_k c_k$. Thus

$$(*) \qquad \sum_{k=1}^{n} a_k^2 \leq \sum_{k=1}^{n} b_k^2 + \sum_{k=1}^{n} c_k^2 + 2\sum_{k=1}^{m} b_k c_k.$$

Now by Lemma I.15, we have

$$\left(\sum_{k=1}^{n} b_k c_k\right)^2 \leq \sum_{k=1}^{n} b_k^2 \sum_{k=1}^{n} c_k^2.$$

Take the square root of both sides to obtain

$$\left(\sum_{k=1}^{n} b_k c_k\right) \leq \sqrt{\sum_{k=1}^{n} b_k^2 \sum_{k=1}^{n} c_k^2}.$$

Combine this with inequality (*) to obtain

$$\sum_{k=1}^{n} a_k^2 \leq \sum_{k=1}^{n} b_k^2 + \sum_{k=1}^{n} c_k^2 + 2\sqrt{\sum_{k=1}^{n} b_k^2 \sum_{k=1}^{n} c_k^2}$$

Taking the square root of both sides produces the result. □

*Proof of Theorem I.12.* The positivity of $\rho$ is clear from the use of positive square root in the definition, and the symmetry is given by the symmetry of the metric on the constituent spaces. Thus is suffices to demonstrate the triangle inequality. Let $a_k = \rho(x_k, z_k)$, $b_k = \rho(x_k, y_k)$, and $c_k = \rho(y_k, z_k)$. By the triangle inequality in the constituent spaces, we have $a_k \leq b_k + c_k$ for $i = 1, \ldots, n$. Apply Lemma I.16 to obtain the result. $\square$

**Example I.17.** Let $(X, \rho)$ be a metric space, and let $X^k$ denote the cartesian product of $k$ copies of $X$, endowed with the product metric.

**Example I.18.** Let $\mathbb{R}^k$ denote the cartesian product of $k$ copies of the real line. The product metric on $\mathbb{R}^k$ as defined in Example I.4 is the same as that defined in Theorem I.12.

**Example I.19.** Let $\mathbb{S}^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2\}$ be the unit circle, considered as a metric subspace of $\mathbb{R}^2$. Let $\mathbb{T} = \mathbb{S}^1 \times \mathbb{S}^1$, endowed with the product metric. Then $\mathbb{T}$ is a torus.

CHAPTER II

# Metric Topology

ABSTRACT. The chapter discusses bounded sets, open sets, closed sets, and neighborhoods in a metric space.

## 1. Bounded Sets

**Definition II.1.** Let $(X, \rho)$ be a metric space, and let $A \subset X$. Define the *diameter* of $A$ with respect to $\rho$ to be

$$\operatorname{diam}(A) = \sup\{\rho(a, b) \mid a, b \in A\};$$

by convention, the diameter of an empty set is zero. Note that $\operatorname{diam}(A)$ is an extended real number which can be $\infty$.

We say that $A$ is *bounded* if $\operatorname{diam}(A) < \infty$.

**Proposition II.2.** *Let $(X, \rho)$ be a metric space, and let $A, B \subset X$. Then*

**(a)** $\operatorname{diam}(A) = 0 \Leftrightarrow |A| \leq 1$*;*
**(b)** $A \subset B \Rightarrow \operatorname{diam}(A) \leq \operatorname{diam}(B)$*;*
**(c)** $A \cap B \neq \varnothing \Rightarrow \operatorname{diam}(A \cup B) \leq \operatorname{diam}(A) + \operatorname{diam}(B)$*.*

*Proof.* Recall that $|A|$ is the *cardinality* of $A$, and is defined to be the number of elements in $A$. If $A$ contains at least two distinct elements, the distance between them is positive, so the diameter of $A$ is greater than 0. On the other hand, if $A$ contains exactly one element, say $A = \{a\}$, then $\{\rho(a, b) \mid a, b \in A\} = \{\rho(a, a)\} = \{0\}$, and the supremum of this set is zero.

Suppose $A \subset B \subset X$. Set $S_A = \{\rho(a_1, a_2) \mid a_1, a_2 \in A\}$, and $S_B = \{\rho(b_1, b_2) \mid b_1, b_2 \in B\}$. Clearly $S_A \subset S_B$, so $\operatorname{diam}(A) = \sup(S_A) \leq \sup(S_B) = \operatorname{diam}(B)$.

Finally, suppose that $A, B \subset X$ and that $A \cap B \neq \varnothing$. Suppose that $\operatorname{diam}(A \cup B) > \operatorname{diam}(A) + \operatorname{diam}(B)$, and let $\epsilon = \frac{1}{2}(\operatorname{diam}(A \cup B) - (\operatorname{diam}(A) + \operatorname{diam}(B)))$. Then, from the definition of diameter, there exist points $c_1, c_2 \in A \cup B$ such that $\operatorname{diam}(A \cup B) - \rho(c_1, c_2) > \epsilon$. ☐

**Exercise II.1.** Let $(X, \rho)$ be a metric space, and let $G = \operatorname{diam}(A)$ with respect to $\rho$. Define a function

$$\widehat{\rho} : X \times X \to \mathbb{R} \quad \text{by} \quad \widehat{\rho}(x, y) = \frac{\rho(x, y)}{1 + \rho(x, y)}.$$

**(a)** Show that $\widehat{\rho}$ is a metric on $X$.

Let $H = \operatorname{diam}(X)$ with respect to $\widehat{\rho}$.

**(b)** Show that $H \leq 1$.
**(c)** Show that if $G = \infty$, then $H = 1$.
**(d)** Show that if $X$ is finite, then $H = \frac{G}{1+G}$.

## 2. Open Sets

**Definition II.3.** Let $(X, \rho)$ be a metric space. Let $x_0 \in X$ and let $\delta > 0$. Set

$$B(x_0, \delta) = \{x \in X \mid \rho(x, x_0) < \delta\};$$

this is known as an *open ball about* $x_0$ *of radius* $\delta$.

Let $U \subset X$. We say that $U$ is *open* if

$$\forall u \in U \exists \delta > 0 \mid B(x_0, \delta) \subset U.$$

**Proposition II.4.** *Let $(X, \rho)$ be a metric space, and let $A \subset X$. Then $A$ is open if and only if $A$ can be expressed as a union of open balls.*

*Proof.* Suppose that $A$ is open; then for every $a \in A$ there exists $\delta_a > 0$ such that $B(a, \delta_a) \subset A$. Then

$$A = \cup_{a \in A} B(a, \delta_a),$$

so $A$ is a union of open balls.

On the other hand, suppose that $A$ is the union of open balls. Let $a \in A$. Then $a \in B(x, \delta)$ for some $x \in X$ and $\delta > 0$, where $B(x, \delta) \subset A$. Then $B(a, \delta - \rho(x, a)) \subset B(x, \delta)$. To see this, let $b \in B(a, \delta - \rho(x, a))$. Then the triangle inequality implies that

$$\rho(x, b) \leq \rho(x, a) + \rho(a, b) \leq \rho(x, a) + (\delta - \rho(x, a)) = \delta.$$

Thus $B(a, \delta) \subset B(x, \delta) \subset A$, and $A$ satisfies the definition of an open set. □

**Proposition II.5.** *Let $(X, \rho)$ be a metric space. Then*

- **(a)** *The sets $\varnothing$ and $X$ are open.*
- **(b)** *The union of any collection of open subsets of $X$ is open.*
- **(c)** *The intersection of any finite collection of open subsets of $X$ is open.*

*Proof.* The empty set vacuously satisfies the condition for openness; every $x \in \varnothing$ has an open ball contained in $\varnothing$, because there is no $x \in \varnothing$. If $x \in X$, then $B(x, 1) \subset X$ by definition of $B(x, 1)$.

Suppose that $\{U_\alpha \mid \alpha \in I\}$ is a collection of open subsets of $X$ indexed by the indexing set $I$. Let $U = \cup_{\alpha \in I} U_\alpha$. Let $x \in U$. Then $x \in U_\alpha$ for some $\alpha \in I$. Since $U_\alpha$ is open, $B(x, \delta) \subset U_\alpha$ for some $\delta > 0$. Then $B(x, \delta) \subset U$, since $U_\alpha \subset U$. Thus $U$ is open.

Suppose that $\{U_1, \ldots, U_n\}$ is a finite collection of open subsets of $X$. Let $U = \cap_{i=1}^n U_i$, and let $x \in U$. Then $x \in U_i$ for $i = 1, \ldots, n$. Since each of these is open, there exist positive real number $\delta_1, \ldots, \delta_n$ such that $x \in B(x, \delta_i)$ for $i = 1, \ldots, n$.

Set $\delta = \min\{\delta_1, \ldots, \delta_n\}$. Then $B(x, \delta) \subset U_i$ for $i = 1, \ldots, n$. Thus $B(x, \delta) \subset \cap_{i=1}^n U_i = U$. In this way, we see that $U$ is open. □

**Definition II.6.** Let $X$ be a set. A *topology* on $X$ is a collection of subsets $\mathcal{T} \subset \mathcal{P}(X)$ satisfying

- **(T1)** $\varnothing \in \mathcal{T}$ and $X \in \mathcal{T}$;
- **(T2)** if $\mathcal{U} \subset \mathcal{T}$, then $\cup \mathcal{U} \in \mathcal{T}$;
- **(T3)** if $\mathcal{U} \subset \mathcal{T}$ and $\mathcal{U}$ is finite, then $\cap \mathcal{U} \in \mathcal{T}$.

The elements of $\mathcal{T}$ are called *open sets*. The pair $(X, \mathcal{T})$ is called a *topological space*.

*Observation* II.1. If $(X, \rho)$ is a metric space, then the collection of open subsets of $X$ is a topology on $X$.

## 3. Closed Sets

**Definition II.7.** Let $(X, \rho)$ be a metric space. Let $F \subset X$. We say that $F$ is *closed* if $X \smallsetminus F$ is open.

*Warning* II.1. Just because a set is not open does not mean that it is closed. For example, $[1, 2) \subset \mathbb{R}$ is neither.

**Proposition II.8.** *Let $(X, \rho)$ be a metric space. Then*

- **(a)** *The sets $\varnothing$ and $X$ are closed.*
- **(b)** *The intersection of any collection of closed subsets of $X$ is closed.*
- **(c)** *The union of any collection of closed subsets of $X$ is closed.*

*Proof.* Recall *DeMorgan's Laws*, which state that the union of complements is the complement of the intersection, and the intersection of complements is the complement of the union. Then Proposition II.8 follows from Proposition II.5 and DeMorgan's Laws. $\square$

## 4. Interior, Closure, and Boundary

**Definition II.9.** Let $(X, \rho)$ be a metric space, and let $A \subset X$. The *interior* of $A$ is the union of all open subsets of $A$:

$$A^\circ = \bigcap_{\substack{U \subset A \\ U \text{ is open}}} U.$$

Since the union of open sets is open, this is clearly the largest open subset of $A$.

**Proposition II.10.** *Let $(X, \rho)$ be a metric space and let $A \subset X$. Then $A$ is open if and only if $A = A^\circ$.*

**Definition II.11.** Let $(X, \rho)$ be a metric space, and let $A \subset X$. The *closure* of $A$ is the intersection of all closed subsets of $X$ which contain $A$:

$$\overline{A} = \bigcap_{\substack{A \subset F \\ F \text{ is closed}}} F.$$

Since the intersection of closed sets is closed, this is clearly the smallest closed subset of $X$ which contains $A$.

**Proposition II.12.** *Let $(X, \rho)$ be a metric space and let $A \subset X$. Then $A$ is closed if and only if $A = \overline{A}$.*

**Definition II.13.** Let $(X, \rho)$ be a metric space and let $A \subset X$. The *boundary* of $A$ is $\partial A = \overline{A} \smallsetminus A^\circ$.

## 5. Neighborhoods

**Definition II.14.** Let $(X, \rho)$ be a metric space and let $x \in X$. A *basic open neighborhood* of $x$ is an open ball of the form $B(x, \delta)$ for some $\delta > 0$. An *open neighborhood* of $x$ is any open subset of $X$ which contains $x$. A *neighborhood* of $x$ is any subset of $X$ which contains an open neighborhood of $x$.

**Exercise II.2.** Let $(X, \rho)$ be a metric space. Let $x \in X$ and let $A, B \subset X$ be neighborhoods of $x$. Show that $A \cap B$ is a neighborhood of $x$.

**Definition II.15.** If $A$ and $B$ are sets, we say that $A$ *intersects* $B$ if $A \cap B \neq \varnothing$.

Let $(X, \rho)$ be a metric space. Let $A \subset X$ and $x \in X$.

We say that $x$ is an *interior point* of $A$ if there exists a neighborhood of $x$ which is contained in $A$.

We say that $x$ is a *closure point* of $A$ if for every neighborhood of $x$ intersects $A$.

We say that $x$ is a *boundary point* of $A$ if for every neighborhood of $x$ intersects both $A$ and $X \smallsetminus A$.

**Proposition II.16.** *Let $(X, \rho)$ be a metric space. Let $A \subset X$ and $x \in X$. Then*

**(a)** *$x$ is an interior point of $A$ if and only if $x \in A^\circ$;*
**(b)** *$x$ is a closure point of $A$ if and only if $x \in \overline{A}$;*
**(c)** *$x$ is a boundary point of $A$ if and only if $x \in \partial A$.*

**Definition II.17.** Let $(X, \rho)$ be a metric space. Let $A \subset X$ and let $x \in X$.

A *deleted neighborhood* of $x$ is a subset $V \subset X$ such that $V = U \smallsetminus \{x\}$ for some neighborhood $U$ of $x$.

We say that $x$ is an *isolated point* of $A$ if every deleted neighborhood of $x$ is contained in $X \smallsetminus A$.

We say that $x$ is an *accumulation point* of $A$ if every deleted neighborhood of $x$ intersects $A$.

**Proposition II.18.** *Let $(X, \rho)$ be a metric space. Let $A \subset X$. Set*

$$B = \{x \in X \mid x \text{ is an isolated point of } A\};$$

$$C = \{x \in X \mid x \text{ is an accumulation point of } A\}.$$

*Then $\overline{A} = B \cup C$.*

CHAPTER III

# Completeness

ABSTRACT. This chapter discusses sequences, subsequences, bounded sequences, and Cauchy sequences. In the process, the Bolzano-Weierstrass property and the completeness property of metric spaces are discussed. We show that these properties of a metric space carry over to products.

## 1. Sequences

**Definition III.1.** Let $X$ be a set. A *sequence* in $X$ is a function $a : \mathbb{N} \to X$. We write $a_n$ instead of $a(n)$, and we write $(a_n)_{n\in\mathbb{N}}$ or simply $(a_n)$ to denote the entire sequence.

One can think of a sequence as an ordered tuple with infinity many entries; hence the notation.

**Definition III.2.** Let $(X, \rho)$ be a metric space and let $(a_n)$ be a sequence in $X$. Let $p \in X$. We say that $(a_n)$ *converges to* $p$, and write $\lim_{n\to\infty} a_n = p$, if

$$\forall\epsilon > 0\ \exists N \in \mathbb{N} \mid n \geq N \Rightarrow \rho(a_n, p) < \epsilon.$$

If $(a_n)$ converges to $p$, we call $p$ a *limit point* of $(a_n)$.

**Definition III.3.** Let $(X, \rho)$ be a metric space and let $(a_n)$ be a sequence in $X$. Let $q \in X$. We say that $(a_n)$ *clusters at* $q$ if

$$\forall\epsilon > 0\ \forall N \in \mathbb{N}\ \exists n \geq N \mid \rho(a_n, q) < \epsilon.$$

If $(a_n)$ clusters at $q$, we call $q$ a *cluster point* of $(a_n)$.

**Example III.4.** Let $X = \mathbb{R}$ and $\rho(x, y) = |x - y|$. Then our new definitions for convergence and clustering become identical to our previous definitions for this particular case.

**Exercise III.1.** Let $\mathbb{S}^1$ be the unit circle together with the subspace metric inherited from $\mathbb{R}^2$. Let $(a_n)$ be the sequence in $\mathbb{S}^1$ defined by

$$a_n = \left(\cos\frac{2\pi n}{6}, \sin\frac{2\pi n}{6}\right).$$

Find the cluster points of $(a_n)$.

**Exercise III.2.** Let $X$ be a set and define a metric $\rho$ on $X$ by

$$\rho(x,y) = \begin{cases} 0 & \text{if } x = y; \\ 1 & \text{otherwise .} \end{cases}$$

Let $(a_n)$ be a sequence in $X$.

**(a)** Show that $p \in X$ is a limit point of $(a_n)$ if and only if

$$\exists N \in \mathbb{N} \mid n \geq N \Rightarrow a_n = p.$$

**(b)** Show that $q \in X$ is a cluster point of $(a_n)$ if and only if

$$\forall N \in \mathbb{N} \exists n \geq N \mid a_n = q.$$

**Definition III.5.** Let $(X, \rho)$ be a metric space and let $(a_n)$ be a sequence from $X$. For each $N \in \mathbb{N}$, the $N^{\text{th}}$ *tail* of $(a_n)$ is defined to be the set

$$\{a_n \mid n \geq N\} = \{x \in X \mid x = a_n \text{ for some } n \geq N\}.$$

**Proposition III.6.** *Let $(X, \rho)$ be a metric space, $(a_n)$ a sequence from $X$, and $p \in X$. Then the following conditions are equivalent:*
**(L1)** *For every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $n \geq N \Rightarrow \rho(a_n, p) < \epsilon$.*
**(L2)** *For every neighborhood $U$ of $p$ there exists $N \in \mathbb{N}$ such that $n \geq N \Rightarrow a_n \in U$.*
**(L3)** *Every neighborhood of $p$ contains a tail of $(a_n)$.*
**(L4)** *Every neighborhood of $p$ contains $a_n$ for all but finitely many $n \in \mathbb{N}$.*

*Proof.*

(**L1** $\Rightarrow$ **L2**) Suppose that for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $n \geq N \Rightarrow \rho(a_n, p) < \epsilon$. Let $U$ be a neighborhood of $p$. Then there exists $\epsilon > 0$ such that $B(p, \epsilon) \subset U$. Let $N$ be so large that $\rho(a_n, p) < \epsilon$ whenever $n \geq N$. Then for $n \geq N$, we have $a_n \in B(p, \epsilon) \subset U$.

(**L2** $\Rightarrow$ **L3**) Suppose that for every neighborhood $U$ of $p$ there exists $N \in \mathbb{N}$ such that $n \geq N \Rightarrow a_n \in U$. Let $U$ be a neighborhood of $p$ and let $N$ be so large that $n \geq N \Rightarrow a_n \in U$. Then $\{a_n \mid n \geq N\} \subset U$, so $U$ contains the $N^{\text{th}}$ tail of $(a_n)$.

(**L3** $\Rightarrow$ **L4**) Suppose that every neighborhood $U$ of $p$ contains a tail of $(a_n)$. Let $U$ be a neighborhood of $p$ and let $N \in \mathbb{N}$ such that $\{a_n \mid n \geq N\} \subset U$. If $a_n \notin U$ for some $n \in \mathbb{N}$, then $a_n \notin \{a_n \mid n \geq N\}$, so $n < N$. There are only finitely many such $n$.

(**L4** $\Rightarrow$ **L1**) Suppose that every neighborhood of $p$ contains $a_n$ for all but finitely many $n$. Let $\epsilon > 0$. Then $B(p, \epsilon)$ is a neighborhood of $p$, so $a_n \in B(p, \epsilon)$ for all but finitely many $n \in \mathbb{N}$. Let $N = 1 + \max\{n \in \mathbb{N} \mid a_n \notin B(p, \epsilon)\}$. Then for $n > N$, we have $\rho(a_n, p) < \epsilon$. $\square$

**Proposition III.7.** *Let $(X, \rho)$ be a metric space, $(a_n)$ a sequence from $X$, and $q \in X$. Then the following conditions are equivalent:*
**(C1)** *For every $\epsilon > 0$ and every $N \in \mathbb{N}$ there exists $n \geq N$ such that $\rho(a_n, q) < \epsilon$.*
**(C2)** *For every neighborhood $U$ of $q$ and every $N \in \mathbb{N}$ there exists $n \geq N$ such that $a_n \in U$.*
**(C3)** *Every neighborhood of $q$ intersects every tail of $(a_n)$.*
**(C4)** *Every neighborhood of $q$ contains $a_n$ for infinitely many $n \in \mathbb{N}$.*

*Proof.*
(**C1** $\Rightarrow$ **C2**) Suppose that for every $\epsilon > 0$ and every $N \in \mathbb{N}$ there exists $n \geq N$ such that $\rho(a_n, q) < \epsilon$. Let $U$ be a neighborhood of $q$ and let $N \in \mathbb{N}$. Then there exists $\epsilon > 0$ such that $B(p, \epsilon) \subset U$; thus there exists $n \geq N$ such that $\rho(a_n, q) < \epsilon$. But this says that $a_n \in B(q, \epsilon)$, so $a_n \in U$.

(**C2** $\Rightarrow$ **C3**) Suppose that for every neighborhood $U$ of $q$ and every $N \in \mathbb{N}$ there exists $n > N$ such that $a_n \in U$. Let $U$ be a neighborhood of $q$ and let $\{a_n \mid n \geq N\}$ be an arbitrary tail of $(a_n)$. Then for some $n \geq N$, we have $a_n \in U$. But $a_n \in \{a_n \mid n \geq N\}$, so $a_n \in \{a_n \mid n \geq N\} \cap U$, and $\{a_n \mid n \geq N\}$ intersects $U$.

(**C3** $\Rightarrow$ **C4**) Suppose that every neighborhood of $q$ intersects every tail of $(a_n)$. Let $U$ be a neighborhood of $q$. Suppose bwoc that $U$ contains $a_n$ for only finitely many $n \in \mathbb{N}$. Let $m$ be the largest natural number such that $a_m \in U$. Then $[a_n : m+1]$ is a tail of $(a_n)$ which does not intersect $U$; this is a contradiction.

(**C4** $\Rightarrow$ **C1**) Suppose that every neighborhood of $q$ contains $a_n$ for infinitely many $n \in \mathbb{N}$. Let $\epsilon > 0$ and $N \in \mathbb{N}$. Then $U = B(q, \epsilon)$ is a neighborhood of $q$, and $U$ contains $a_n$ for infinitely many $n \in \mathbb{N}$. One such $n$ must be larger than $N$; if $n \in \mathbb{N}$ such that $a_n \in U$, then $\rho(a_n, q) < \epsilon$. □

**Proposition III.8.** *Let $(X, \rho)$ be a metric space, $(a_n)$ a sequence from $X$, and $p \in X$. If $(a_n)$ converges to $p$, then $(a_n)$ clusters at $p$, and $p$ is the only cluster point.*

*Proof.* Suppose that $(a_n)$ converges to $p$. Then every neighborhood of $p$ contains $a_n$ for all but finitely many $n$. Thus there are infinitely many $n$ such that $a_n$ is in the neighborhood. By Proposition III.7 **(d)**, $(a_n)$ clusters at $p$.

To see that $p$ is the only cluster point, let $q \in X$, $q \neq p$; we show that $(a_n)$ does not cluster at $q$. Let $\epsilon = \frac{\rho(p,q)}{2}$ and let $U = B(p, \epsilon)$ and $V = B(q, \epsilon)$. Then $U$ and $V$ are disjoint neighborhoods of $p$ and $q$ respectively.

Let $A$ be a tail of $(a_n)$ such that $A \subset U$. Since $U \cap V = \varnothing$, we have $A \cap V = \varnothing$, so $V$ is a neighborhood of $q$ which does not intersect $A$. Thus $(a_n)$ does not cluster at $q$, by III.7 **(c)**. □

**Exercise III.3.** Find an example of a sequence $(a_n)$ of real numbers and a real number $q \in \mathbb{R}$ such that $(a_n)$ clusters at $q$ but does not converge to $q$.

## 2. Subsequences

**Definition III.9.** Let $(X, \rho)$ be a metric space and let $(a_n)$ be a sequence in $X$, where $a : \mathbb{N} \to X$ is the function defining $(a_n)$. A *subsequence* of $(a_n)$ is the composition $a \circ n$ of $a$ with a strictly increasing sequence $n : \mathbb{N} \to \mathbb{N}$ of positive integers. Let $n_k = n(k)$, and denote the subsequence by $(a_{n_k})$.

**Proposition III.10.** *Let $(X, \rho)$ be a metric space and let $(a_n)$ be a sequence in $X$, Then $q \in X$ is a cluster point of $(a_n)$ if and only if $(a_n)$ has a subsequence which converges to $q$.*

## 3. Bounded Sequences

**Definition III.11.** Let $(X, \rho)$ be a metric space and let $(a_n)$ be a sequence in $X$. We say that $(a_n)$ is *bounded* if the set $\{a_n \mid n \in \mathbb{N}\}$ is a bounded set.

**Exercise III.4.** Let $(X, \rho)$ be a metric space and let $(a_n)$ be a sequence in $X$. Then $(a_n)$ is bounded if and only if there exists a point $c \in X$ and a positive real number $R > 0$ such that $\rho(a_n, c) \leq R$ for all $n \in \mathbb{N}$.

**Definition III.12.** Let $(X, \rho)$ be a metric space. We say that $X$ has the *Bolzano-Weierstrass property* if every bounded sequence in $X$ has a convergent subsequence.

**Example III.13.** We have already shown that $\mathbb{R}$ has the Bolzano-Weierstrass property.

**Proposition III.14.** *Let $(X, \rho)$ be a metric space. Then $X$ has the Bolzano-Weierstrass property if and only if every sequence has a cluster point.*

*Proof.* This follows immediately from Proposition III.10. □

**Proposition III.15.** *Let $(X, \rho)$ be a metric space. Then $X$ has the Bolzano-Weierstrass property if and only if every bounded infinite subset of $X$ has an accumulation point.*

*Proof.* Suppose that $X$ has the Bolzano-Weierstrass property. Then every bounded sequence in $X$ has a cluster point. Let $A \subset X$ be a bounded infinite set. Since $A$ is infinite, there exists an injective function $a : \mathbb{N} \to A$. This produces a sequence $(a_n)$. This sequence is bounded, so it has a cluster point, say $q \in X$.

We claim that $q$ is an accumulation point of $A$. To see this, let $U$ be a neighborhood of $q$. Since $q$ is a cluster point, $U$ contains $a_n$ for infinitely many $n$. Since $a$ is injective, at $a_n = q$ for at most one $n$. Thus $U \smallsetminus \{q\}$ contains $a_n$ for some $n$, and $a_n \in A$. Thus $U$ intersects $A$, and $q$ is a cluster point.

Suppose that every bounded infinite subset of $X$ has an accumulation point. Let $(a_n)$ be a sequence in $A$. Let $B = \{a_n \mid n \in \mathbb{N}\}$. If $B$ is finite, then there exists $b \in B$ such that $b = a_n$ for infinitely many $n$. In this case, $b$ is a cluster point of $A$. On the other hand, if $B$ is infinite, it has an accumulation point, and this accumulation point will be a cluster point of $(a_n)$. □

## 4. Cauchy Sequences

**Definition III.16.** Let $(X, \rho)$ be a metric space and let $(a_n)$ be a sequence in $X$. We say that $(a_n)$ is a *Cauchy sequence* if

$$\forall \epsilon > 0 \exists N \in \mathbb{N} \mid m, n \geq N \Rightarrow \rho(a_m, a_n) < \epsilon.$$

**Definition III.17.** Let $(X, \rho)$ be a metric space. We say that $X$ is *complete* if every Cauchy sequence in $X$ converges.

This definition of completeness appears different than the completeness axiom which we use to obtain the reals from the rationals. We now relabel that definition.

**Definition III.18.** Let $S$ be an ordered set. We say that $S$ has the *supremum property* if every subset of $S$ which is bounded above has a least upper bound. We say that $S$ has the *infimum property* if every subset of $S$ which is bounded below has a greatest lower bound.

We have already shown that a sequence in $\mathbb{R}$ converges if and only if it is a Cauchy sequence. We now show that for subsets of $\mathbb{R}$, the supremum and infimum properties are equivalent to the new completeness property; in this way, the new definition is a generalization of the old one.

**Proposition III.19.** *Let $A \subset \mathbb{R}$. Then $A$ is a complete metric subspace of $\mathbb{R}$ if and only if $A$ has the supremum and infimum properties.*

*Proof.* Suppose that $A$ is a complete metric subspace of $\mathbb{R}$. Then every Cauchy sequence in $A$ converges to a point in $A$. Let $B \subset A$ be bounded above; Then $B$ has a supremum in the reals, say $x = \sup B$. Then for each $n \in \mathbb{N}$, there exists $b_n \in B$ such that $x - b_n < \frac{1}{2^n}$. Then for $m < n$, we have $|b_n - b_m| \frac{1}{2^n}$. Therefore $(b_n)$ is a Cauchy sequence, which converges to a point in $A$. But clearly $\lim b_n = a$, so $\sup B = x \in A$. Similarly, $B$ has the infimum property.

On the other hand, suppose that $A$ has the supremum and infimum properties, and let $(a_n)$ be a Cauchy sequence in $A$. Then $(a_n)$ converges in $\mathbb{R}$, say to $x \in \mathbb{R}$. Let $u_n = \inf\{a_m \mid m \geq n\}$. Since $A$ has the infimum property, $u_n \in A$ for every $n \in \mathbb{N}$. Also, $(u_n)$ is an increasing sequence which converges to $x$, so $x = \sup\{u_n \mid n \in \mathbb{N}\}$. Since $A$ has the supremum property, this is also in $A$. Thus every Cauchy sequence in $A$ converges to a point in $A$. □

**Proposition III.20.** *Let $(X, \rho)$ be a metric space and let $(a_n)$ be a Cauchy sequence in $X$. Then $(a_n)$ is bounded.*

*Proof.* Since $(a_n)$ is a Cauchy sequence, there exists $N \in \mathbb{N}$ such that $m, n \geq N$ implies $\rho(a_m, a_n) < 1$. Let $M = \max\{\rho(a_i, a_N) \mid i < N\} \cup \{1\}$. Then $\rho(a_n, a_N) < M$ for every $n \in \mathbb{N}$. □

**Proposition III.21.** *Let $(X, \rho)$ be a metric space and let $(a_n)$ be a Cauchy sequence in $X$. If $(a_n)$ has a subsequence converging to $p \in X$, then $(a_n)$ converges to $p$.*

*Proof.* Suppose that $(a_{n_k})$ is a subsequence of $(a_n)$ which converges to $p \in X$. Let $\epsilon > 0$, and let $K$ be so large that $k \geq K$ implies that $\rho(a_{n_k}, p) < \frac{\epsilon}{2}$. Let $M$ be so large that $m, n \geq M$ implies $\rho(a_m, a_n) < \frac{\epsilon}{2}$. Let $N = \max\{K, M\}$. Then for $n \geq N$, we have

$$\rho(a_n, p) \leq \rho(a_n, a_N) + \rho(a_N, p) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Therefore $(a_n)$ converges to $p$. □

**Proposition III.22.** *Let $(X, \rho)$ be a metric space. If $X$ has the Bolzano-Weierstrass property, then $X$ is complete.*

*Proof.* Suppose that $X$ has the Bolzano-Weierstrass property, and let $(a_n)$ be a Cauchy sequence. By Proposition III.20, $(a_n)$ is bounded, and so has a convergent subsequence. By Proposition III.21, $(a_n)$ converges. Thus $X$ is complete. □

We have seen that if a metric space has the Bolzano-Weierstrass property, then it is complete. One may conjecture that these properties are equivalent. The following counterexample shows this is not the case.

**Example III.23.** Let $X$ be any set any consider the discrete metric on $X$ such that the distance between distinct points equals 1. In this space, Cauchy sequences are eventually constant, and so they converge. Thus $X$ is complete. However, *every* sequence in $X$ is bounded, so $X$ has the Bolzano-Weierstrass property if and only if $X$ is finite.

Next we would like to show the following propositions.

**Proposition III.24.** *A sequence converges in $\mathbb{R}^k$ if and only if each of the coordinate sequences converges. A sequence is Cauchy in $\mathbb{R}^k$ if and only if each of the coordinate sequences is Cauchy. The metric space $\mathbb{R}^k$ is complete.*

**Proposition III.25. Bolzano-Weierstrass Theorem**
*Every bounded sequence in $\mathbb{R}^k$ has a convergent subsequence.*

*Discussion.* Proposition III.24 is a lemma for Proposition III.25, which is a generalization of the Bolzano-Weierstruass Theorem which we have already shown for $\mathbb{R}$ (the case $k = 1$). However, these propositions can be generalized even further, and we postpone the proofs for this more general context, which we take up next. □

## 5. Product Space Sequences

**Proposition III.26.** *Let $(X_1, \rho_1), \ldots, (X_k, \rho_k)$ be a finite collection of metric spaces. Let $X = \times_{i=1}^{k} X_i$, and let $\rho : X \times X \to \mathbb{R}$ be the product metric on $X$. Then*

- **(a)** *A sequence is bounded in $X$ if and only if each of the coordinate sequences is bounded.*
- **(b)** *A sequence converges in $X$ if and only if each of the coordinate sequences converges.*
- **(c)** *A sequence is Cauchy in $X$ if and only if each of the coordinate sequences is Cauchy.*

**(d)** *The metric space $X$ is complete if and only if each of the spaces $X_i$ is complete.*
**(e)** *The metric space $X$ has the Bolzano-Weierstrass property if and only if each of the spaces $X_i$ has the Bolzano-Weierstrass property.*

*Preliminary Observation.* Now suppose that $x = (x_1, \ldots, x_k)$ and $y = (y_1, \ldots, y_k)$ are points in $X$, where $x_j, y_j \in X_i$. Observe that, since all metrics are positive, we have

$$\rho_j(x_j, y_j) \leq \sqrt{\sum_{i=1}^{k} \rho(x_i, y_i)} = \rho(x, y) \leq \sqrt{k} \max\{\rho(x_i, y_i) \mid i = 1, \ldots, k\}.$$

□

*Notation.* A point in $X$ is an $k$-tuple with entries for $X_1$ through $X_k$. If we denote these entries with subscripts, we must find another place to indicate the position of such an $k$-tuple in a sequence. Thus let $(x^{(n)})$ denote a sequence in $X$, where

$$x^{(n)} = (x_1^{(n)}, \ldots, x_k^{(n)}),$$

where $x_i^{(n)} \in X_i$. □

*Proof of* **(a)**. This follows from the observation. □

*Proof of* **(b)**. Suppose that $(x_i^{(n)})$ converges for $i = 1, \ldots, k$, say to $L_i \in X_i$. Let $L = (L_1, \ldots, L_k)$. Let $\epsilon > 0$. Let $N$ be so large that $\rho_i(x_i^{(n)}, L_i^{(n)}) < \frac{1}{k}\epsilon^2$ for $n \geq N$. Then for $n \geq N$ we have

$$\rho(x_n, L) = \sqrt{\sum_{i=1}^{k} \rho(x_i^{(n)}, L_i)} < \sqrt{\sum_{i=1}^{k} \frac{1}{k}\epsilon^2} = \sqrt{k(\frac{1}{k}\epsilon^2)} = \epsilon.$$

Therefore $\lim x^{(n)} = L$, and in particular, $(x^{(n)})$ converges.

Suppose that $(x^{(n)})$ converges, say to $L = (L_1, \ldots, L_k)$. Let $\epsilon > 0$ and let $n$ be so large that $\rho(x^{(n)}, L) < \epsilon$ for $n \geq N$. Then for $i$ between 1 and $k$ we have

$$\rho_i(x_i^{(n)}, L_i) \leq \rho(x^{(n)}, L) < \epsilon.$$

Thus $\lim_{n\to\infty} x_i^{(n)} = L_i$, and in particular, the sequence $(x_i^{(n)})$ converges. □

*Proof of* **(c)**. Suppose that $(x_i^{(n)})$ is a Cauchy sequence for $i = 1, \ldots, k$. Let $\epsilon > 0$ and let $N$ be so large that $m, n \geq N$ implies

$$\rho_i(x_i^{(m)}, x_i^{(n)}) < \frac{\epsilon}{\sqrt{k}}$$

for all $i = 1, \ldots, k$. Then by the observation, we have

$$\rho(x^{(m)}, x^{(n)}) \leq \epsilon.$$

Suppose that $(x^{(n)})$ is a Cauchy sequence. Let $\epsilon > 0$. Let $N$ be so large that $m, n \geq N$ implies $\rho(x^{(m)}, x^{(n)}) < \epsilon$. Then for $m, n \geq N$, we have

$$\rho_i(x_i^{(m)}, x_i^{(n)}) \leq \rho(x^{(m)}, x^{(n)}) < \epsilon,$$

we says that the coordinate sequence $(x_i^{(n)})$ is a Cauchy sequence. □

*Proof of* **(d)**. We know that a metric space is complete if and only if each of its Cauchy sequences converges.

Suppose each space $(X_i, \rho_i)$ is complete, and consider a Cauchy sequence in $X$. Each of the coordinate sequences are Cauchy by part **(b)**, so each converges since $X_i$ is complete. Then the original sequence converges by part **(a)**, so $X$ is complete.

On the other hand, suppose that $(X, \rho)$ is complete, and let $i \in \{1, \ldots, k\}$. Consider a Cauchy sequence in $X_i$. Construct a sequence in $X$ by selecting a constant $a_i \in X_i$ in every coordinate other than the $i^{\text{th}}$. These are all Cauchy sequences in the coordinate spaces, so the construct sequence in $X$ converges. Thus the original sequence in $X_i$ converges, and $X_i$ is complete. □

*Proof of* **(e)**. Suppose that $X_i$ has the Bolzano-Weierstrass property for $i = 1, \ldots, k$. Then each bounded sequence in $X_i$ has a convergent subsequence. Given a bounded sequence in $X$, each of the coordinate sequences is bounded, and has a convergent subsequence. Select a convergent subsequence $X_1$ for the first coordinate subsequence, and take the corresponding subsequence in $X$. Now select a convergent subsequence in $X_2$ for the second coordinate subsequence of the new sequence in $X$, and again take the corresponding subsequence in $X$. Continue this process $k$ times, and arrive at a sequence in $X$ such that every subsequence converges. This sequence is a subsequence of the original sequence in $X$, and it converges. Thus $X$ has the Bolzano-Weierstrass property.

Suppose that $X$ has the Bolzano-Weierstrass property. Let $i \in \{1, \ldots, k\}$ and let consider a bounded sequence in $X_i$. Construct a sequence in $X$ by selecting a constant $a_i \in X_i$ in every coordinate other than the $i^{\text{th}}$. This is bounded in $X$, and so has a convergent subsequence. The $i^{\text{th}}$ coordinate sequence of this subsequence converges in $X_i$, and is a subsequence of the original sequence in $X_i$. Thus $X_i$ has the Bolzano-Weierstrass property. □

**Corollary III.27.** *The space* $\mathbb{R}^k$ *is complete and has the Bolzano-Weierstrass property.*

**Example III.28.** Consider $\mathbb{R}^\infty$, whose points are all infinite tuples of real numbers with all but finitely many entries equal to zero. Construct a sequence $(x^{(n)})$ in $\mathbb{R}^\infty$ by setting

$$x_i^{(n)} = \begin{cases} 1 & \text{if } i = n; \\ 0 & \text{otherwise.} \end{cases} \tag{$\dagger$}$$

Then $(x^{(n)})$ is bounded (it is completely contained inside the closed unit ball), yet has no convergent subsequence. Thus $\mathbb{R}^\infty$ does not have the Bolzano-Weierstrass property. Note that the sequence above is not a Cauchy sequence.

However, consider this example. Construct a sequence $(y^{(n)})$ in $\mathbb{R}^\infty$ by setting

$$y_i^{(n)} = \begin{cases} \frac{1}{2^i} & \text{if } i \leq n; \\ 0 & \text{otherwise.} \end{cases}$$

This is a Cauchy sequence in $\mathbb{R}^\infty$ which does not converge in $\mathbb{R}^\infty$. So this space is not complete.

**Example III.29.** Let $\ell^2$ be the space of sequences $(x_n)$ in $\mathbb{R}$ with the convergence criterion $\sum_{i=1}^\infty x_i^2 < 0$. Then $\mathbb{R}^\infty$ is a subspace of $\ell^2$, and the sequence ($\dagger$) from Example III.28 does not have a convergent subsequence in $\ell^2$.

However, $\ell^2$ is complete. To show this, proceed as follows. Consider a Cauchy sequence $(x_i^{(n)})$ in $\ell^2$. Show that the coordinate sequences are Cauchy, and so the converge in $\mathbb{R}$; say that $(x_i^{(n)})$ converges to $x_i$ for each $i$. Next see that the sequence $(x_i)$ is in $\ell^2$.

Clearly there is some relationship between the Bolzano-Weierstrass property and completeness. We need the concept of *compactness* to illuminate this further.

CHAPTER IV

# Continuity

## 1. Continuous Functions

### 1.1. Continuity at a Point.

**Definition IV.1.** Let $(X, \rho)$ and $(Y, \tau)$ be metric spaces. Let $f : X \to Y$ and let $a \in X$. We say that $f$ is *continuous at $a$* if

$$\forall \epsilon > 0 \; \exists \delta > 0 \mid \rho(x, a) < \delta \Rightarrow \tau(f(x), f(a)) < \epsilon.$$

**Example IV.2.** Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = 3x - 7$ and let $a \in \mathbb{R}$. Show that $f$ is continuous at $a$.

*Solution.* Let $\epsilon > 0$ and let $\delta = \frac{\epsilon}{3}$. Then

$$|x-a| < \delta \Rightarrow |x-a| < \frac{\epsilon}{3} \Rightarrow |3x-3a| < \epsilon \Rightarrow |3x-7-(3a-7)| < \epsilon \Rightarrow |f(x)-f(a)| < \epsilon.$$

□

### 1.2. Sequential Continuity at a Point.

**Definition IV.3.** Let $(X, \rho)$ and $(Y, \tau)$ be metric spaces. Let $f : X \to Y$ and let $a \in X$. We say that $f$ is *sequentially continuous at $a$* if for every sequence $(x_n)$ in $X$ converging to $a$, the sequence $(f(x_n))$ in $Y$ converges to $f(a)$.

**Proposition IV.4.** *Let $(X, \rho)$ and $(Y, \tau)$ be metric spaces. Let $f : X \to Y$ and let $a \in X$. Then $f$ is continuous at $a$ if and only if $f$ is sequentially continuous at $a$.*

*Proof.* We prove both directions of this implication.

($\Rightarrow$) Suppose that $f$ is continuous at $a$. Let $(x_n)$ be a sequence in $X$ which converges to $a$; we wish to show that $(x_n)$ converges to $f(a)$. Let $\epsilon > 0$. Since $f$ is continuous at $a$, there exists $\delta > 0$ such that $\rho(x, a) < \delta$ implies $\tau(f(x), f(a)) < \epsilon$. Let $N$ be so large that $n \geq N$ implies $\rho(x, a) < \delta$. Then, for $n \geq N$, we have $\tau(f(x), f(a)) < \epsilon$.

($\Leftarrow$) Suppose that $f$ is not continuous at $a$. Then

$$\exists \epsilon > 0 \mid \forall \delta > 0 \; \exists x \in X, \rho(x, a) < \delta \mid \tau(f(x), f(a)) \geq \epsilon.$$

Let $\epsilon$ satisfy the above condition, and for $n \in \mathbb{N}$, let $x_n \in X$ be such that $\rho(x, a) < \frac{1}{n}$, but $\tau(f(x_n), f(a)) \geq \epsilon$. Then $(x_n)$ converges to $a$, but $f(x_n)$ does not converge to $f(a)$. Therefore, $f$ is not sequentially continuous at $a$. □

### 1.3. Topological Continuity at a Point.

**Definition IV.5.** Let $(X, \rho)$ and $(Y, \tau)$ be metric spaces. Let $f : X \to Y$ and let $a \in X$. We say that $f$ is *topologically continuous at* $a$ if for every open neighborhood $V$ of $f(a)$ there exists an open neighborhood $U$ of $a$ such that $f(U) \subset V$.

*Observation* IV.1. Let $(X, \rho)$ and $(Y, \tau)$ be metric spaces. Let $f : X \to Y$ and let $a \in X$. The following conditions are equivalent:

- **(a)** $\rho(x, a) < \delta \Rightarrow \tau(f(x), f(a)) < \epsilon$;
- **(b)** $x \in B(a, \delta) \Rightarrow f(x) \in B(f(a), \epsilon)$;
- **(c)** $f(B(a, \delta)) \subset B(f(a), \epsilon)$.

**Proposition IV.6.** *Let $(X, \rho)$ and $(Y, \tau)$ be metric spaces. Let $f : X \to Y$ and let $a \in X$. Then $f$ is continuous at $a$ if and only if $f$ is topologically continuous at $a$.*

*Proof.* We prove both directions.

($\Rightarrow$) Suppose that $f$ is continuous at $a$, and let $V$ be an open neighborhood of $f(a)$. Then there exists $\epsilon > 0$ such that $B(f(a), \epsilon) \subset V$. Since $f$ is continuous, there exists $\delta > 0$ such that $f(B(a, \delta) \subset B(f(a), \epsilon)$. Let $U = B(a, \delta)$; then $f(U) \subset V$.

($\Leftarrow$) Suppose that $f$ is topologically continuous at $a$. Let $\epsilon > 0$ and let $V = B(f(a), \epsilon)$. Then $V$ is an open neighborhood of $f(a)$ in $Y$, so there exists an open neighborhood $U$ of $a$ in $X$ such that $f(U) \subset V$. Since $U$ is open and $a \in U$, there exists $\delta > 0$ such that $B(a, \delta) \subset U$. Then $f(B(a, \delta)) \subset B(f(a), \epsilon)$. □

### 1.4. Continuity on a Space.

**Definition IV.7.** Let $(X, \rho)$ and $(Y, \tau)$ be metric spaces. Let $f : X \to Y$ and let $A \in X$. We say that $f$ is *continuous on* $A$ if $f$ is continuous at $a$ for every $a \in A$. If $f$ is continuous on $X$, we say simply that $f$ is *continuous.*

**Example IV.8.** Let $f(x) = \frac{3x-2}{x^2-7x+10}$. The natural real domain of this function is $X = \mathbb{R} \smallsetminus \{2, 5\}$. Thus $f : X \to \mathbb{R}$, and $f$ is continuous. In fact, rational functions are always continuous on their domains.

**Definition IV.9.** Let $(X, \rho)$ and $(Y, \tau)$ be metric spaces. Let $f : X \to Y$. We say that $f$ is *topologically continuous* if for every open set $V$ in $Y$, $f^{-1}(V)$ is open in $X$.

**Proposition IV.10.** *Let $(X, \rho)$ and $(Y, \tau)$ be metric spaces. Let $f : X \to Y$. Then $f$ is continuous if and only if $f$ is topologically continuous.*

*Proof.* We prove both directions.

($\Rightarrow$) Suppose that $f$ is continuous, and let $V \subset Y$ be open. Let $U = f^{-1}(V)$, and let $u \in U$. We wish to show that their is an open neighborhood of $u$ contained in $U$. Since $V$ is open, there exists $\epsilon > 0$ such that $B(f(u), \epsilon) \subset V$. Since $f$ is continuous, there exists $\delta > 0$ such that $\rho(x, u) < \delta$ implies $\tau(f(x), f(u)) < \epsilon$, that is, $x \in B(u, \delta)$ implies that $f(x) \in B(f(u), \epsilon) \subset V$, so that $f(B(u, \delta)) \subset V$. Thus $B(u, \delta) \subset U$, so $U$ is open. Therefore $f$ is topologically continuous.

($\Leftarrow$) Suppose that $f$ is topologically continuous. Let $a \in X$ and let $\epsilon > 0$. Let $V = B(f(a), \epsilon)$; then $V$ is open in $Y$, so $UY = f^{-1}(V)$ is open in $X$. Clearly $a \in U$; thus there exists $\delta > 0$ such that $B(a, \delta) \subset U$, and $f(B(a, \delta)) \subset B(f(a), \epsilon)$. This says that $\rho(x, a) < \delta$ implies $\tau(f(x), f(a)) < \epsilon$. □

### 1.5. Composition of Continuous Functions.

**Proposition IV.11.** *Let $(X, \rho)$, $(Y, \tau)$, and $(Z, \chi)$ be metric spaces. Let $f : X \to Y$ be continuous at $a \in X$ and let $g : Y \to Z$ be continuous at $f(a) \in Y$. Then $g \circ f$ is continuous at $a$.*

*Proof.* Let $W$ be an open neighborhood of $g(f(a))$. Since $g$ is continuous, there exists an open neighborhood $V$ of $f(a)$ such that $g(V) \subset W$. Also, since $f$ is continuous, there exists an open neighborhood $U$ of $a$ such that $f(U) \subset V$. Then $g(f(U)) \subset g(V) \subset W$. □

### 1.6. Real-Valued Functions.

**Proposition IV.12.** *Let $(X, \rho)$ be a metric space, and let $f : X \to \mathbb{R}$ be a real-valued function defined on $X$. Let $Y = \{x \in X \mid g(x) \neq 0\}$ and let $k \in \mathbb{R}$. Define new functions as follows:*

- $|f| : X \to \mathbb{R}$ *is defined by* $|f|(x) = |f(x)|$.
- $kf : X \to \mathbb{R}$ *is defined by* $(kf)(x) = kf(x)$;
- $\frac{1}{f} : Y \to \mathbb{R}$ *is defined by* $(\frac{1}{f})(x) = \frac{1}{f(x)}$;

*If $f$ is continuous at $a$, then $kf$ and $moduf$ are continuous at $a$. If $a \in Y$, then $\frac{1}{f}$ is continuous at $a$.*

*Proof.* We show, for example, that $kf$ is continuous at $a \in X$. Let $(x_n)$ be a sequence in $X$ that converges to $a$. Then $\lim kf(x_n) = k \lim f(x_n) = kf(a)$. This shows that $kf$ is continuous at $a$. □

**Proposition IV.13.** *Let $(X, \rho)$ be a metric space, and let $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ be real-valued functions defined on $X$. Define new functions as follows:*

- $f + g : X \to \mathbb{R}$ *is defined by* $(f + g)(x) = f(x) + g(x)$;
- $fg : X \to \mathbb{R}$ *is defined by* $(fg)(x) = f(x) \cdot g(x)$;
- $\max(f, g) : X \to \mathbb{R}$ *is defined by* $\max(f, g)(x) = \max\{f(x), g(x)\}$;
- $\min(f, g) : X \to \mathbb{R}$ *is defined by* $\min(f, g)(x) = \min\{f(x), g(x)\}$;

*If $f$ and $g$ are continuous at $a$, then the above functions are continuous at $a$ (except when dividing by zero).*

*Proof.* We show, for example, that $f + g$ is continuous at $a \in X$. Let $(x_n)$ be a sequence in $X$ that converges to $a$. Then $\lim(f + g)(x_n) = \lim(f(x_n) + g(x_n)) = \lim f(x_n) + \lim g(x_n) = f(a) + g(a) = (f + g)(a)$. This shows that $f + g$ is continuous at $a$. □

## 2. Continuity Examples for $f : \mathbb{R} \to \mathbb{R}$

**Example IV.14.** Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2$. Let $x_0 = 2$. Show that $f$ is continuous at $x_0$.

*Proof.* Let $\epsilon > 0$; we may assume that $\epsilon < 4$. Let $\delta = \sqrt{x_0^2 + \epsilon} - x_0 = \sqrt{4 + \epsilon} - 2$. Thus $(\delta + 2)^2 = 4 + \epsilon$, so $\epsilon = \delta^2 + 4\delta$.

Suppose that $x \in (2 - \delta, 2 + \delta)$. Then $x + 2 < \delta + 4$, and

$$|f(x) - f(x_0)| = |x^2 - 4| = |x - 2|(x + 2) < \delta(4 + \delta) = \epsilon.$$

□

**Example IV.15.** Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^3$. Show that $f$ is continuous.

*Proof.* Let $x_0 \in \mathbb{R}$ and let $\epsilon > 0$. We wish to find $\delta > 0$ such that if $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \epsilon$.

For simplicity, assume that $x_0 > 0$. Let $\delta = \sqrt[3]{x_0^3 + \epsilon} - x_0$. Solving for $\epsilon$ yields $\epsilon = (x_0 + \delta)^3 - x_0^3$.

Let $x \in (x_0 - \delta, x_0 + \delta)$. Then $x > 0$, and

$$\begin{aligned}|f(x) - f(x_0)| &= |x^3 - x_0^3| \\ &= |x - x_0|(x^2 + x_0 x + x_0^2) \\ &< \delta((x_0 + \delta)^2 + x_0(x_0 + \delta) + x_0^2) \\ &= \delta(x_0^2 + 2x_0\delta + \delta^2 + x_0^2 + x_0\delta + x_0^2) \\ &= \delta(3x_0^2 + 3x_0\delta + \delta^2) \\ &= \epsilon.\end{aligned}$$

□

**Example IV.16.** Let $f : [0, \infty) \to \mathbb{R}$ be given by $f(x) = \sqrt{x}$. Show that $f$ is continuous.

*Motivation.* Graph the curve $f(x) = \sqrt{x}$. Select arbitrary $x_0 \in \mathrm{dom}(f)$. Project up and to the right to find the point $\sqrt{x_0}$ on the $y$-axis. Draw an $\epsilon$-band around this point. Project the intersection of this band with the graph of $f$ onto the $x$-axis. Notice that the point on the left of this projection is closer to $x_0$ than is the point on the right. Let $\delta$ be one half of the distance between $x_0$ and the left endpoint of the inverse image of $[f(x_0) - \epsilon, f(x_0) + \epsilon]$. □

*Proof.* Let $x_0 \in [0, \infty)$ and let $\epsilon > 0$; wlog assume that $\epsilon^2 \le x_0$. If $x_0 = 0$, let $\delta = \epsilon^2$; clearly this will work. Otherwise set

$$\delta = \frac{1}{2}(x_0 - (\sqrt{x_0} - \epsilon)^2);$$

this is positive. Note that for $x \in \mathbb{R}$, $|x - x_0| = |\sqrt{x} - \sqrt{x_0}|(\sqrt{x} + \sqrt{x_0})$. Then if $|x - x_0| < \delta$, we have

$$\begin{aligned}|\sqrt{x} - \sqrt{x_0}| &< \frac{\delta}{\sqrt{x} + \sqrt{x_0}} \\ &= \frac{x_0 - (x_0 - 2\sqrt{x_0}\epsilon + \epsilon^2)}{2(\sqrt{x} + \sqrt{x_0})} \\ &= \frac{\epsilon(2\sqrt{x_0} - \epsilon)}{2(\sqrt{x} + \sqrt{x_0})} \\ &< \epsilon\frac{(2\sqrt{x_0} - \epsilon)}{2\sqrt{x_0}} \\ &= \epsilon\left(1 - \frac{\epsilon}{2\sqrt{x_0}}\right) \\ &< \epsilon.\end{aligned}$$

□

**Example IV.17.** Show that every polynomial function is continuous.

*Proof.* This is tedious but obviously important. We build it gradually.

*Claim 1:* The constant function $f(x) = C$, where $C \in \mathbb{R}$, is continuous.
Let $x_0 \in \mathbb{R}$ and let $\epsilon > 0$. Set $\delta = 1$. Then if $|x - x_0| < \delta$, we have $|f(x) - f(x_0)| = 0 < \epsilon$. Thus $f$ is continuous in this case.

*Claim 2:* The identity function $f(x) = x$ is continuous.
Let $x_0 \in \mathbb{R}$ and let $\epsilon > 0$. Set $\delta = \epsilon$. Then if $|x - x_0| < \delta$, we have $|f(x) - f(x_0)| = |x - x_0| < \delta = \epsilon$, so $f$ is continuous in this case.

*Claim 3:* The function $f(x) = x^n$ is continuous.
By induction on $n$. For $n = 1$, the function $g(x) = x$ is the identity function, and so it is continuous. By induction, $h(x) = x^{n-1}$ is continuous. Then by the Continuous Arithmetic Proposition, $f = gh$ is continuous in this case.

*Claim 4:* The monomial function $f(x) = a_n x^n$ is continuous, where $a_n \in \mathbb{R}$ is constant.
By Claim 1, $g(x) = a_n$ is continuous, and by Claim 3, $h(x) = x^n$ is continuous, so there product $f = gh$ is continuous.

*Claim 5:* The polynomial function $f(x) = a_0 + a_1 x + \cdots + a_n x^n$ is continuous.
By induction on $n$, the degree of the polynomial.

For $n = 0$, $f(x)$ is constant and therefore continuous.

Assume that $g(x) = a_0 + \cdots + a_{n-1}x^{n-1}$ is continuous. By Claim 4, $h(x) = a_n x^n$ is continuous. Then $f = g + h$ is continuous by the Continuous Arithmetic Proposition. □

**Example IV.18.** Show that every rational function is continuous.

*Proof.* Let $f$ be a rational function. Then $f(x) = p(x)/q(x)$, where $p$ and $q$ are polynomial functions. Since $p$ and $q$ are continuous, then $f$ is continuous on its domain by a Proposition from the arithmetic of continuous functions. □

**Example IV.19.** Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

Show that $f$ is discontinuous at every real number.

*Proof.* Let $x_0 \in \mathbb{R}$. To show that $f$ is discontinuous at $x_0$, it suffices to find $\epsilon > 0$ such that for every $\delta > 0$, there exists $x \in (x_0 - \delta, x_0 + \delta)$ with $|f(x) - f(x_0)| \geq \epsilon$.

Let $\epsilon = \frac{1}{2}$ and let $\delta > 0$. Then there exists both a rational and an irrational in $(x_0 - \delta, x_0 + \delta)$. If $x_0$ is rational, let $x_1$ be an irrational in this interval, and we have $|f(x_1) - f(x_0)| = 1 > \epsilon$; if $x_0$ is irrational, let $x_2$ be a rational in this interval, and we still have $|f(x_2) - f(x_0)| = 1 > \epsilon$. Thus $f$ is not continuous at $x_0$. □

**Example IV.20.** Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

Show that $f$ is continuous at $x = 0$ and discontinuous at all nonzero real numbers.

*Proof.* Let $x_0 \in \mathbb{R} \setminus \{0\}$; we show that $f$ is discontinuous at $x_0$. Let $\epsilon = \frac{|x_0|}{2}$ and let $\delta > 0$. Then there exists both a rational and an irrational in $(x_0 - \delta, x_0 + \delta)$. If $x_0$ is rational, let $x_1$ be an irrational in this interval, and we have $|f(x_1) - f(x_0)| =$

$|x_0| > \epsilon$. If $x_0$ is irrational, let $x_2$ be a rational in this interval such that $|x_2| > |x_0|$ and we still have $|f(x_2) - f(x_0)| = |x_2| > |x_0| > \epsilon$. Thus $f$ is not continuous at $x_0$.

Now we consider the behavior of $f$ at zero. Let $\epsilon > 0$ and let $\delta = \epsilon$. Then if $|x - 0| < \delta$, we have $|f(x) - f(0)| = 0$ if $x$ is irrational and $|f(x) - f(0)| = |x|$ if $x$ is rational; in either case, $|f(x) - f(0)| \leq |x| < \delta = \epsilon$, so $f$ is continuous at zero. □

**Example IV.21.** If $r \in \mathbb{Q}$, there exists $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ such that $r = \frac{p}{q}$. Define $q : \mathbb{Q} \to \mathbb{R}$ by

$$q(r) = \min\{q \in \mathbb{N} \mid r = \frac{p}{q} \text{ for some } p \in \mathbb{Z}\}.$$

Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ \frac{1}{q(x)} & \text{if } x \text{ is rational} \end{cases}$$

Show that $f$ is discontinuous at every rational and continuous at every irrational.

*Proof.* Suppose that $x_0$ is rational. We wish to show that $f$ is not continuous at $x_0$. It suffices to find $\epsilon > 0$ such that for every $\delta > 0$ there exists $x_1 \in (x_0 - \delta, x_0 + \delta)$ with $|x_0 - x_1| > \epsilon$.

Since $x_0$ is rational, we have $x_0 = \frac{p}{q(x_0)}$ for some $p \in \mathbb{Z}$. Let $\epsilon = \frac{1}{2q(x_0)}$ and let $\delta > 0$. Then $(x_0 - \delta, x_0 + \delta)$ contains an irrational number, say $x_1$; then $|x_0 - x_1| < \delta$ but $|f(x_0) - f(x_1)| = \frac{1}{q(r)} > \epsilon$. Thus $f$ cannot be continuous at $x_0$.

Suppose that $x_0$ is irrational. Let $\epsilon > 0$. It suffices to find $\delta > 0$ such that $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon$.

Let $N \in \mathbb{N}$ be so large that $\frac{1}{N} < \epsilon$. Let $a$ be the greatest integer which is less than $x_0$ and $b$ be the least integer which is greater than $x_0$; then $b = a + 1$ and $x_0 \in [a, b]$.

For $q \in \mathbb{Q}$, there exist only finitely many points in the set $[a, b] \cap \{\frac{k}{q} \mid k \in \mathbb{Z}\}$ (in fact, this set contains no more than $q$ points). Thus the set

$$D = [a, b] \cap \{\frac{k}{q} \mid k \in \mathbb{Z}, q \leq N\}$$

is finite (there are no more than $\frac{N(N+1)}{2}$ points in this set). Let

$$\delta = \min\{|x_0 - d| \mid d \in D\};$$

since this set is a finite set of positive real numbers, the minimum exists as a positive real number. Then $(x_0 - \delta, x_0 + \delta) \subset [a, b]$. Let $x \in (x_0 - \delta, x_0 + \delta)$. If $x$ is irrational, we have $|f(x) - f(x_0)| = 0 < \epsilon$, and if $x$ is rational, we have $|f(x) - f(x_0)| = \frac{1}{q(x)} < \frac{1}{N} < \epsilon$. Thus $f$ is continuous at $x_0$. □

## 3. Isometries,Contractions, and Homeomorphisms

### 3.1. Isometries.

**Definition IV.22.** Let $(X, \rho)$ and $(Y, \tau)$ be metric spaces. Let $f : X \to Y$. We say that $f$ *preserves distance* if

$$\tau(f(a), f(b)) = \rho(a, b).$$

A bijective function which preserves distance is called an *isometry.*
An injective function which preserves distance is called an *isometric embedding.*

**Proposition IV.23.** *Let $(X, \rho)$ and $(Y, \tau)$ be metric spaces. Let $f : X \to Y$ which preserves distance. Then $f$ is an isometric embedding.*

*Proof.* We only have to show that $f$ is injective. Let $a, b \in X$ such that $f(a) = f(b)$. Then $\rho(a, b) = \tau(f(a), f(b)) = 0$, so by property **(M1)** of a metric, $a = b$. Thus $f$ is injective. □

**Example IV.24.** Let $f : \mathbb{R} \to \mathbb{R}$ be an isometry. Then $f(x) = ux + b$ for some $b \in \mathbb{R}$, where $u = \pm 1$.

*Proof.* Let $b = f(0)$. Now let $x \in \mathbb{R}$ and let $y = f(x)$. Now $x = |x - 0| = |f(x) - f(0)| = |y - b|$, so $x = \pm(y - b)$; thus $y = ux + b$, where $u = \pm 1$. It remains to show that $u$ is independent of $x$.

Thus assume that $f(x_1) = x_1 + b$ and $f(x_2) = -x_2 + b$; it suffices to show that $x_1 = 0$ or $x_2 = 0$. Now $|x_1 - x_2| = |f(x_1) - f(x_2)| = |x_1 + b - (-x_2 + b)| = |x_1 + x_2|$. Squaring both sides and cancelling yields $-x_1x_2 = x_1x_2$, so either $x_1 = 0$ or $x_2 = 0$. □

**Example IV.25.** Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be an isometry. Then exactly one of these conditions hold:

**(1)** there exists a line $y = mx + b$ such that $f$ is reflection across this line;
**(2)** there exists a point $(x_0, y_0)$ and an angle $\alpha$ such that $f$ is rotation by $\alpha$ around $(x_0, y_0)$.

**Exercise IV.1.** Describe the isometries of $\mathbb{R}^3$.

**Exercise IV.2.** Let $X = \{(\cos\alpha, \sin\alpha) \in \mathbb{R}^2 \mid \alpha = \frac{2\pi}{n} \text{ for some } n \in \mathbb{Z}\}$. Describe the isometries of $X$.

### 3.2. Contractions.

**Definition IV.26.** Let $(X, \rho)$ and $(Y, \tau)$ be metric spaces. Let $f : X \to Y$. We say that $f$ is a *contraction* if there exists $M > 0$ such that

$$\tau(f(a), f(b)) \leq M\rho(a, b)$$

for all $a, b \in X$.

**Example IV.27.** An isometric embedding is a contraction.

**Proposition IV.28.** *Let $(X, \rho)$ and $(Y, \tau)$ be metric spaces. Let $f : X \to Y$ be a contraction. Then $f$ is continuous.*

*Proof.* Since $f$ is a contraction, there exists $M > 0$ such that $\tau(f(a), f(b)) \leq M\rho(a, b)$ for every $a, b \in X$.

Let $a \in X$ and let $\epsilon > 0$. Let $\delta = \frac{\epsilon}{M}$. Then

$$\rho(a, b) < \delta \Rightarrow \rho(a, b) < \frac{\epsilon}{M} \Rightarrow M\rho(a, b) < \epsilon \Rightarrow \tau(f(a), f(b)) < \epsilon.$$

□

### 3.3. Homeomorphisms.

**Definition IV.29.** Let $(X, \rho)$ and $(Y, \tau)$ be metric spaces. Let $f : X \to Y$. We say that $f$ is a *homeomorphism* if $f$ is a continous bijective function whose inverse is also bijective.

It is natural to suppose that a continuous bijective function always has a continuous inverse. This is not the case.

**Example IV.30.** Let $X = (0, 1) \cup [2, 3)$ and let $Y = (0, 2)$. Define $f : X \to Y$ by

$$f(x) = \begin{cases} x & \text{if } x \in (0, 1); \\ x - 1 & \text{if } x \in [2, 3). \end{cases}$$

This function is clearly bijective and continuous at every point in $X$; however, its inverse is discontinuous.

**Example IV.31.** Let $X = \mathbb{R}$. Let $\rho$ be the standard metric on $X$ and let $\tau$ be the discrete metric. Then $\mathrm{id} : (X, \tau) \to (X, \rho)$ is bijective and continuous, but the inverse is not continuous.

**Example IV.32.** Let $X = (-\frac{\pi}{2}, \frac{\pi}{2})$ and $Y = \mathbb{R}$, endowed with the usual metric. Let $f : X \to Y$ be given by $f(x) = \tan x$. Then $f$ is bijective and continuous, and its inverse is $f^{-1}(x) = \arctan x$. Thus a homeomorphism can map a bounded space onto an unbounded space.

**Example IV.33.** Let $(X, \rho)$ be a metric space, and define

$$\widehat{\rho} : X \times X \to \mathbb{R} \quad \text{by} \quad \widehat{\rho}(x, y) = \frac{\rho(x, y)}{1 + \rho(x, y)}.$$

We have seen that $(X, \widehat{\rho})$ is a metric space. View the identity map $\mathrm{id}_X : X \to X$ as a function from $(X, \rho)$ to $(X, \widehat{\rho})$. Then $\mathrm{id}_X$ is a bijective contraction, and its inverse is also continuous. Thus $\mathrm{id}_X$ is a homeomorphism.

**Definition IV.34.** Let $X$ be a set and let $\rho$ and $\tau$ be metrics on $X$. We say that $\rho$ and $\tau$ are *equivalent* if they produce the same open sets.

**Definition IV.35.** Let $(X, \rho)$ be a metric space. The *topology induced by $\rho$ on $X$* is

$$\mathcal{T}_{(X,\rho)} = \{U \subset X \mid U \text{ is open in } X\}.$$

**Proposition IV.36.** *Let $X$ be a set and let $\rho$ and $\tau$ be metrics on $X$. The following conditions are equivalent:*

- **(a)** $\mathrm{id} : (X, \rho) \to (X, \tau)$ *is a homeomorphism;*
- **(b)** $\mathcal{T}_{(X,\rho)} = \mathcal{T}_{(X,\tau)}$*;*
- **(c)** *every open ball with respect to one metric contains an open ball with respect to the other metric.*

*If any of these conditions hold, we say that the metrics $\rho$ and $\tau$ are* equivalent.

## 4. Projections and Injections

### 4.1. Open Maps.

**Definition IV.37.** Let $(X, \rho)$ and $(Y, \tau)$ be metric spaces, and let $f : X \to Y$ be a function.

We say that $f$ is an *open map* if for every open set $U$ in $X$, $f(U)$ is open in $Y$.

**Example IV.38.** Let $(X, \rho)$ be any metric space, and let $(Y, \tau)$ be a discrete metric space. Then every function $f : (X, \rho) \to (Y, \tau)$ is an open map.

**Example IV.39.** The function $\sin : \mathbb{R} \to \mathbb{R}$ is a continuous function which sends the open set $(0, 4\pi)$ to the closed interval $[0, 1]$. Thus sin is not an open map.

**Example IV.40.** Let $\mathbb{S}^1$ denote the unit circle in the Euclidean plane, endowed with the subspace metric. The function $f : \mathbb{R} \to \mathbb{S}^1$ given by $f(\theta) = (\cos\theta, \sin\theta)$ is an open continuous map which is not a homeomorphism.

### 4.2. Projection.

**Definition IV.41.** Let $(X_1, \rho_1), \ldots, (X_k, \rho_k)$ be metric spaces. Let $X = \times_{i=1}^k X_i$, endowed with the product metric. Let $j \in \{1, \ldots, k\}$.

The $i^{\text{th}}$ *projection map* is the function

$$p_j : X \to X_j \quad \text{given by } p_j(x_1, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_k) = x_j.$$

**Proposition IV.42.** *Let $(X_1, \rho_1), \ldots, (X_k, \rho_k)$ be metric spaces. Let $X = \times_{i=1}^k X_i$, endowed with the product metric $\rho$. Let $j \in \{1, \ldots, k\}$. Let $a = (a_1, \ldots, a_j, \ldots, a_k) \in X$. Then $p_j(B(a, \delta)) = B(a_j, \delta)$.*

*Proof.* We show containment in both directions.
($\subset$) Let $x_j \in p_j(B(a, \delta))$; then $x_j = p_j(x)$ for some $x = (x_1, \ldots, x_{i-1}, x_j, x_{i+1}, \ldots, x_k)$ with $x \in B(a, \delta)$; that is, $\rho(x, a) < \delta$. Now

$$\rho_j(a_j, x_j) \le \sqrt{\sum_{i=1}^k \rho_i(x_i, a_i)^2} = \rho(a, x) < \delta.$$

So $x_j \in B(a_j, \delta)$.
($\supset$) Let $x_j \in B(a_j, \delta)$, and set $x = (a_1, \ldots, a_{i-1}, x_j, a_{i+1}, \ldots, a_k)$. One sees that

$$\rho(a, x) = \sqrt{\rho_j(a_j, x_j)^2} = \rho_j(a_j, x_j) < \delta.$$

Thus $x \in B(a, \delta)$, so $x_j = p_j(x) \in p_j(B(a, \delta))$. □

**Proposition IV.43.** *Let $(X_1, \rho_1), \ldots, (X_k, \rho_k)$ be metric spaces. Let $X = \times_{i=1}^k X_i$, endowed with the product metric $\rho$. Let $j \in \{1, \ldots, k\}$. Then $p_j : X \to X_j$ is a continuous open map.*

*Proof.* Let $a = (a_1, \ldots, a_j, \ldots, a_k) \in X$, so that $p_j(a) = a_j$. Let $\epsilon > 0$, and let $\delta = \epsilon$. Then $p_j(B(a, \delta)) \subset B(a_j, \epsilon) = B(p_j(a), \epsilon)$. Thus $p_j$ is continuous at $a$.

Now suppose that $U$ is open in $X$, and let $a_j \in p_j(U)$. Then $a_j = p_j(a)$ for some $a = (a_1, \ldots, a_j, \ldots, a_k) \in U$. Since $U$ is open, there exists $\delta > 0$ such that $B(a, \delta) \subset U$. By the previous proposition, $B(a_j, \delta) = p_j(B(a, \delta)) \subset p_j(U)$. Thus $p_j(U)$ is open. □

### 4.3. Injection.

**Definition IV.44.** Let $(X_1, \rho_1), \dots, (X_k, \rho_k)$ be metric spaces. Let $X = \times_{i=1}^k X_i$, endowed with the product metric. Let $j \in \{1, \dots, k\}$, and select a point $a = (a_1, \dots, a_k) \in X$.

The $j^{\text{th}}$ *injection map with respect to* $a$ is the function

$$q_j : X_j \to X \qquad \text{given by } q_j(x_j) = (a_1, \dots, a_{j-1}, x_j, a_{j+1}, \dots, a_k).$$

**Proposition IV.45.** *Let $(X_1, \rho_1), \dots, (X_k, \rho_k)$ be metric spaces. Let $X = \times_{i=1}^k X_i$, endowed with the product metric. Let $j \in \{1, \dots, k\}$, and select a point $a = (a_1, \dots, a_k) \in X$. Let $p_j$ be projection and $q_i$ be injection with respect to $a$. Then $p_j \circ q_j = \mathrm{id}_{X_j}$.*

*Proof.* This is clear. □

**Proposition IV.46.** *Let $(X_1, \rho_1), \dots, (X_k, \rho_k)$ be metric spaces. Let $X = \times_{i=1}^k X_i$, endowed with the product metric. Let $i \in \{1, \dots, k\}$, and select a point $a = (a_1, \dots, a_k) \in X$. Then $q_i : X_i \to X$ is a continuous function.*

*Proof.* Let $a = (a_1, \dots, a_k) \in X$; then $q_j(a_j) = a.$ , and $\rho(x, a) = \rho_j(x_j, a_j)$. Let $\epsilon > 0$, and set $\delta = \epsilon$. Let $x_j \in B(a_j, \delta)$, and let $x = \rho_j(x_j)$. Then $\rho(x, a) = \rho_j(x_j, a_j)$, so

$$\rho(x, a) = \rho_j(x_j, a_j) < \delta = \epsilon,$$

which shows that $q_j$ is continuous at $a$. □

**Exercise IV.3.** Show that projection is a contraction.

**Exercise IV.4.** Show that injection is an isometric embedding.

**Definition IV.47.** Closed map.

**Exercise IV.5.** Show that injection is a closed map.

CHAPTER V

# Compactness

ABSTRACT. This chapter discusses the concept of compactness. We prove the Heine-Borel Theorem for $\mathbb{R}$, and show that the Heine-Borel property is inherited for finite products of metric spaces.

## 1. Compactness

**Definition V.1.** Let $(X, \rho)$ be a metric space and let $A \subset X$.

A *cover* of $A$ is a collection of subsets $\mathcal{C} \subset \mathcal{P}(X)$ such that $A \subset \cup\mathcal{C}$.

Let $\mathcal{C}$ be a cover of $A$. We say that $\mathcal{C}$ is a *finite cover* if $A$ is a finite set. We say that $\mathcal{C}$ is an *open cover* if the elements of $\mathcal{C}$ are open sets. A *subcover* of $\mathcal{C}$ is a subset $\mathcal{D} \subset \mathcal{C}$ such that $\mathcal{D}$ is itself a cover of $A$.

We say that $A$ is *compact* if every open cover of $A$ has a finite subcover.

*Remark* V.1. Notice that in the phrase "finite open cover", the word "finite" applies to the cover itself, whereas the word "open" applies to the subsets of $X$ in the cover.

**Example V.2.** Let $X = \mathbb{R}$ and $A = \mathbb{Z}$. Let $I_n = (n - \frac{1}{3}, n + \frac{1}{3})$. Let $\mathcal{C} = \{I_n \mid n \in \mathbb{Z}\}$. Then $\mathcal{C}$ is an open cover of $\mathbb{Z}$ with no finite subcover. Thus $\mathbb{Z}$ is not compact.

**Example V.3.** Let $X = \mathbb{R}$ and $A = (0, 1)$. Let $I_n = (0, 1 - \frac{1}{n})$. Let $\mathcal{C} = \{I_n \mid n \in \mathbb{N}\}$. Then $\mathcal{C}$ is an open cover of $(0, 1)$ with no finite subcover. Thus $(0, 1)$ is not compact.

**Proposition V.4.** *Let $(X, \rho)$ be a metric space and let $A = \{a_1, \ldots, a_n\} \subset X$ be a finite subset. Then $A$ is compact.*

*Proof.* Let $\mathcal{C}$ be an open cover of $A$. Then for each $a_i \in A$, there exists and open set $U_i \in \mathcal{C}$ such that $a_i \in U_i$. Then $A \subset \cup_{i=1}^n U_i$, and $\{U_1, \ldots, U_n\}$ is a finite subcover of $\mathcal{C}$. Thus $A$ is compact. □

**Proposition V.5.** *Let $a, b \in \mathbb{R}$ with $a < b$. Then the closed interval $[a, b] \subset \mathbb{R}$ is compact.*

*Proof.* Let $\mathcal{C}$ be an open cover of $[a, b]$.

Let $x \in [a, b]$ and let $U_x \in \mathcal{C}$ be an open set which contains $x$. Then there exists $\epsilon_x > 0$ such that $(x - \epsilon_x, x + \epsilon_x) \subset U_x$. Let

$$B = \{x \in [a, b] \mid [a, x] \text{ can be covered by a finite subcover of } \mathcal{C}\}.$$

Note that $B$ is nonempty, since the closed interval $[a, a + \frac{\epsilon_a}{2}] \subset U_a$, and $\{U_a\}$ is a finite subcover of $\mathcal{C}$, so for example $a + \frac{\epsilon_a}{2} \in B$.

Let $z = \sup B$; clearly $a + \frac{\epsilon_a}{2} \leq z \leq b$. We claim that $z \in B$, and that $z = b$. To see this, let $\epsilon = \min\{\epsilon_z, z - a\}$. Then $z - \frac{\epsilon}{2} \in B$. Let $\mathcal{D}$ be a finite subcover of $\mathcal{C}$ which covers $[a, z - \frac{\epsilon}{2}]$, and let $\mathcal{E} = \mathcal{D} \cup \{U_z\}$. Then $\mathcal{E}$ is finite and covers $[a, z]$, so $z \in B$.

Now suppose that $z < b$, and set $\delta = \min\{\epsilon, z - b\}$. Then $z < z + \frac{\delta}{2} < b$, and $\mathcal{E}$ covers $[a, z + \frac{\delta}{2}]$; since $z + \frac{\delta}{2} \in [a, b]$, this contradicts the definition of $z$. Thus $z = b$. This completes the proof. □

## 2. Properties of Compactness

The first proposition says that all compact subsets of a metric space are closed and bounded.

**Proposition V.6.** *Let $(X, \rho)$ be a metric space, and let $K \subset X$ be a compact set. Then $K$ is closed and bounded.*

*Proof.* Suppose that $K$ is not bounded, and let $a \in K$. Let

$$\mathcal{C} = \{B(a, n) \mid n \in \mathbb{N}\}.$$

This is a cover of $K$ by open sets (it actually covers all of $X$). However, since $K$ is unbounded, $K$ is not contained in $B(a, n)$ for any $n \in \mathbb{N}$. Thus $\mathcal{C}$ has no finite subcover, and $K$ is not compact.

Let $\delta > 0$ and let $b \in X$. Let $D(b, \delta) = \{x \in X \mid \rho(x, b) \leq \delta\}$. We claim that this set is closed; to see this, note that if $a \in X \setminus D(b, \delta)$, then $B(a, \rho(a, b) = \delta) \subset X \setminus D(b, \delta)$.

Suppose that $K$ is compact; we wish to show that $K$ is closed. Thus we show that the complement of $K$ is open. Let $b \in X \setminus K$, and set

$$\mathcal{C} = \{X \setminus D(b, \frac{1}{n}) \mid n \in \mathbb{N}\}.$$

Then $\mathcal{C}$ is an open cover of $K$ (in fact, it covers $X \setminus \{b\}$). Thus it has a finite subcover $\mathcal{D}$. Let $n$ be the largest number such that $X \setminus D(b, \frac{1}{n})$ is in $\mathcal{D}$. Then clearly $K \subset X \setminus D(b, \frac{1}{n})$, so $B(b, \frac{1}{n}) \subset X \setminus K$. Thus $X \setminus K$ is open, so $K$ is closed. □

The next proposition says the a closed subset of a compact set is compact.

**Proposition V.7.** *Let $(X, \rho)$ be a metric space, and let $K \subset X$ be a compact set. Let $F \subset K$. If $F$ is closed, then $F$ is compact.*

*Proof.* Suppose $F$ is closed. Then $U = X \setminus F$ is open. Let $\mathcal{C}$ be an open cover of $F$. Then $\mathcal{C} \cup \{U\}$ is an open cover of $K$, and so it has a finite subcover, say $\mathcal{D}$. Let $\mathcal{E} = \mathcal{D} \setminus \{U\}$. Now $\mathcal{E}$ is a finite subcover of $\mathcal{C}$. □

The next proposition says that the continuous image of a compact set is compact.

**Proposition V.8.** *Let $(X, \rho)$ and $(Y, \tau)$ be metric spaces, and let $f : X \to Y$ be a continuous function. If $K \subset X$ is compact, then $f(K)$ is compact.*

*Proof.* Let $\mathcal{V}$ be an open cover of $f(K)$, and set

$$\mathcal{U} = \{U \subset X \mid U = f^{-1}(V) \text{ for some } V \in \mathcal{V}\}.$$

Since $f$ is continuous, $\mathcal{U}$ is a collection of open sets which covers $K$. Thus $\mathcal{U}$ has a finite subcover, say $\{U_1, \dots, U_n\}$. Now for $i = 1, \dots, n$, we have $U_i \in \mathcal{U}$, so $U_i$ is the preimage of some set $V_i \in \mathcal{V}$, so that $V_i = f(U_i)$. Then

$$K \subset \cup_{i=1}^n U_i \ \Rightarrow \ f(K) \subset f(\cup_{i=1}^n U_i) = \cup_{i=1}^n f(U_i) = \cup_{i=1}^n V_i.$$

Thus $\{V_1, \dots, V_n\}$ is a finite subcover of $\mathcal{V}$, and $f(K)$ is compact. □

## 3. Heine-Borel Theorem

**Theorem V.9** (Heine-Borel Theorem for $\mathbb{R}$)**.** *Let $A \subset \mathbb{R}$. Then $A$ is compact if and only if $A$ is closed and bounded.*

*Proof.* The forward direction is true in any metric space, as has been stated as Proposition V.6. Thus we prove that in $\mathbb{R}$, closed and bounded sets are compact.

Suppose that $A$ is closed and bounded; we wish to show that $A$ is compact. Since $A$ is bounded, there exists $M > 0$ such that $A \subset [-M, M]$. The set $[-M, M]$ is a closed interval, and so it is compact by Proposition V.5. Thus $A$ is a closed subset of a compact set, and therefore is compact by Proposition V.7. □

**Proposition V.10.** *Let $K \subset \mathbb{R}$ be a compact. Then* $\inf K \in K$ *and* $\sup K \in K$.

*Proof.* Since $K$ is bounded, then $\sup K$ exists as a real number, say $b = \sup K$. Suppose $b \notin K$; then $\{(-\infty, b - \frac{1}{n}) \mid n \in \mathbb{N}\}$ is an open cover of $K$ with no finite subcover, contradicting that $K$ is compact. Thus $b \in K$. Similarly, $\inf K \in K$. □

**Definition V.11.** Let $(X, \rho)$ be a metric space. We say that $X$ has the *Heine-Borel property* if every closed and bounded subset of $X$ is compact.

Recall these facts:

- **(a)** Projection is continuous;
- **(b)** Projection is an open map;
- **(c)** Projection is a contraction;
- **(d)** Injection is continuous;
- **(e)** Injection is a closed map;
- **(f)** Injection is an isometric embedding.

**Proposition V.12.** *The finite product of compact sets is compact.*

**Proposition V.13.** *Let $(X_1, \rho_1), \ldots, (X_k, \rho_k)$ be a finite collection of metric spaces. Let $X = \times_{i=1}^{k} X_i$, and let $\rho : X \times X \to \mathbb{R}$ be the product metric on $X$. Then $X$ has the Heine-Borel property if and only if if and only if each of the spaces $X_i$ has the Heine-Borel property.*

*Proof.* Suppose that $X$ has the Heine-Borel property, and let $K_i \subset X_i$ be closed and bounded. Then $\iota_i(K_i)$ is closed and bounded, and so $\iota_i(K_i)$ is compact. Therefore $\pi_i(\iota_i(K_i)) = K$ is compact, so $X_i$ has the Heine-Borel property.

Suppose that $X_i$ has the Heine-Borel property for $i = 1, \ldots, k$, and let $K \subset X$ which is closed and bounded.

Then $K_i = \pi_i(K)$ is closed and bounded for each $i$. Thus $K_i$ is compact, so $\times_{i=1}^{k} X_i$ is compact by Proposition V.12. Since $K$ is a closed subset of $\times_{i=1}^{k} X_i$, $K$ is compact by Proposition V.8. Thus $X$ has the Heine-Borel property. □

CHAPTER VI

# Connectedness

Abstract. This chapter discusses the concept of connectedness and its how we may use it to prove the Intermediate Value Theorem.

## 1. Connectedness

**Definition VI.1.** Let $(X, \rho)$ be a metric space and let $A \subset X$. We say that $A$ is *disconnected* if there exist disjoint open sets $U_1$ and $U_2$ such that $A \cap U_1 \neq \varnothing$, $A \cap U_2 \neq \varnothing$, and $A \subset U_1 \cup U_2$. We say that $A$ is *connected* if it is not disconnected.

*Observation* VI.1. Let $I \subset \mathbb{R}$. Then $I$ is an interval if and only if

- if $x_1, x_2 \in I$ and $x_1 < z < x_2$, then $z \in I$.

**Proposition VI.2.** *Let $A \subset \mathbb{R}$. Then $A$ is connected if and only if $A$ is an interval.*

*Proof.* We prove the contrapositive of each implication.

($\Rightarrow$) Suppose that $A$ is not an interval. Then there exist $x_1, x_2 \in A$ and $z \notin A$ such that $x_1 < z < x_2$. If $U_1 = (-\infty, z)$ and $U_2 = (z, \infty)$, then $x_1 \in U_1$, $x_2 \in U_2$, and $A \subset U_1 \cap U_2$. Thus $A$ is not connected.

($\Leftarrow$) Suppose that $A$ is not connected. Then there exist disjoint open sets $U_1, U_2 \subset \mathbb{R}$ such that $A \cap U_1 \neq \varnothing$, $A \cap U_2 \neq \varnothing$, and $A \subset U_1 \cup U_2$. Let $x_1 \in A \cap U_1$ and $x_2 \in A \cap U_2$. Without loss of generality, assume that $x_1 < x_2$. Let $z = \inf\{u \in U_2 \mid u > x_1\}$; then $x_1 \leq z \leq x_2$.

Suppose $z \in U_1$; then a neighborhood of $z$ is contained in $U_1$. But since $z$ is the infimum of a subset of $U_2$, every neighborhood of $z$ intersects $U_2$. This contradicts that $U_1 \cap U_2 = \varnothing$. Thus $z \notin U_1$. In particular, $x_1 < z$.

Suppose that $z \in U_2$; then a neighborhood of $z$ is contained in $U_2$, so there exists $y \in U_2$ with $x_1 < y < z$. This contradicts the definition of $z$. Thus $z \notin U_2$. In particular, $z < x_2$, so $x_1 < z < x_2$.

Since $z \notin U_1 \cup U_2$ but $A \subset U_1 \cup U_2$, $z \notin A$. Thus $A$ is not an interval. □

**Proposition VI.3.** *Let $(X, \rho)$ and $(Y, \tau)$ be metric spaces, and let $f : X \to Y$ be a continuous function. If $A \subset X$ is connected, then $f(A)$ is connected.*

*Proof.* Suppose that $f(A)$ is not connected. Then there exist disjoint open subsets $V_1$ and $V_2$ of $Y$ such that $f(A) \cap V_1 \neq \varnothing$, $f(A) \cap V_2 \neq \varnothing$, and $f(A) \subset V_1 \cup V_2$.

Let $U_1 = f^{-1}(V_1)$ and $U_2 = f^{-1}(V_2)$. Then $A \cap U_1 \neq \varnothing$, $A \cap U_2 \neq \varnothing$, and $A \subset U_1 \cup U_2$. Thus $A$ is disconnected. □

## 2. Intermediate Value Theorem

**Proposition VI.4.** *Let $(X, \rho)$ be a metric space and let $K \subset X$ be compact and connected. Let $f : X \to \mathbb{R}$ be continuous. Then $f(K)$ is a bounded closed interval.*

*Proof.* The image of a compact set is compact, and the compact subsets of $\mathbb{R}$ are closed and bounded.

The image of a connected set is connected, and the connected subsets of $\mathbb{R}$ are intervals.

The result follows. □

**Theorem VI.5** (Intermediate Value Theorem)**.** *Let $f : [a, b] \to \mathbb{R}$ be a continuous function. If $f(a)f(b) < 0$, then there exists $c \in [a, b]$ such that $f(c) = 0$.*

*Proof.* Since $f$ is continuous, the image of $[a, b]$ is a bounded closed interval. Since $f(a)f(b) < 0$, either $f(a) < 0 < f(b)$ or $f(b) < 0 < f(a)$. In either case, 0 is in the image. □

# Solutions

**Exercise VI.5.** Let $\mathcal{F}_{[a,b]}$ denote the set of all bounded functions $f : [a, b] \to \mathbb{R}$. Let $X = \mathcal{F}_{[a,b]}$ and for $f, g \in X$ define

$$\rho(f, g) = \max\{|f(x) - g(x)| \mid x \in [a, b]\}.$$

Show that $(X, \rho)$ is a metric space.

**Exercise VI.6.** Let $\mathcal{C}_{[a,b]}$ denote the set of all continuous functions $f : [a, b] \to \mathbb{R}$. Let $X = \mathcal{C}_{[a,b]}$ and for $f, g \in X$ define

$$\rho(f, g) = \int_a^b |f - g| \, dx.$$

Show that $(X, \rho)$ is a metric space.

*Solution.* In order to prove this, we will need these properties of integration:

**Lemma VI.6.** *Let $f, g : [a, b] \to \mathbb{R}$ be integrable, and let $c \in \mathbb{R}$. Then*

**(a)** $\int_a^b f(x) + g(x) \, dx = \int_a^b f(x) \, dx + \int_a^b g(x) \, dx$;
**(b)** $\int_a^b cf(x) \, dx = c \int_a^b f(x) \, dx$.

**Lemma VI.7.** *Let $f : [a, b] \to [, \infty)$ be a continuous function. If $\int_a^b f(x) \, dx = 0$, then $f(x) = 0$ for every $x \in [a, b]$.*

**Lemma VI.8.** *Let $f, g : [a, b] \to [0, \infty)$ be continuous functions. If $f(x) \leq g(x)$ for every $x \in [a, b]$, then $\int_a^b f(x) \, dx \leq \int_a^b g(x) \, dx$.*

We have $\rho(f, f) = \int_a^b |f - f| \, dx = \int_a^b 0 \, dx = 0$; moreover, Lemma VI.7 tells us that if the integral of a nonnegative continuous function is zero, then that function is the zero function; thus $\rho(f, g) = 0 \Rightarrow |f - g| = 0 \Rightarrow f = g$. Thus **(M1)** follows.

Since $|f - g| = |g - f|$, clearly $\rho(f, g) = \rho(g, f)$. Thus **(M2)** follows.

Let $f, g, h \in \mathcal{C}_[a, b]$. Then $|f(x) - h(x)| \leq |f(x) - g(x)| + |g(x) - h(x)|$ for every $x \in [a, b]$, by the triangle inequality for $\mathbb{R}$. Then by Lemmas VI.8 and VI.6,

$$\begin{aligned}
\rho(f, h) &= \int_a^b |f(x) - h(x)| \, dx \\
&\leq \int_a^b \Big( |f(x) - g(x)| + |g(x) - h(x)| \Big) \, dx \\
&= \int_a^b |f(x) - g(x)| \, dx + \int_a^b |g(x) - h(x)| \, dx \\
&= \rho(f, g) + \rho(g, h).
\end{aligned}$$

□

**Exercise VI.7.** Let $(X, \rho)$ be a metric space, and let $G = \text{diam}(A)$ with respect to $\rho$. Define a function

$$\widehat{\rho} : X \times X \to \mathbb{R} \quad \text{by} \quad \widehat{\rho}(x, y) = \frac{\rho(x, y)}{1 + \rho(x, y)}.$$

**(a)** Show that $\widehat{\rho}$ is a metric on $X$.

Let $H = \text{diam}(X)$ with respect to $\widehat{\rho}$.

**(b)** Show that $H \leq 1$.
**(c)** Show that if $G = \infty$, then $H = 1$.
**(d)** Show that if $X$ is finite, then $H = \frac{G}{1+G}$.

*Solution.* Let $x, y, z \in X$; we wish to show that

$$\widehat{\rho}(x, z) \leq \widehat{\rho}(x, y) + \widehat{\rho}(y, z).$$

Let $a = \rho(x, y)$, $b = \rho(y, z)$ and $c = \rho(x, z)$. Then we wish to show that $a, b, c \geq 0$ and $c \leq a + b$ imply

$$\frac{c}{1+c} \leq \frac{a}{1+a} + \frac{b}{1+b}.$$

Now

$$\begin{aligned}
c \leq a + b &\Rightarrow c \leq (a + b) + (2ab + abc) \quad \text{since } a, b, c \geq 0 \\
&\Rightarrow c + ac + bc + abc \leq (a + b) + (2ab + abc) + (ac + bc + abc) \\
&\Rightarrow c(1 + a + b + ab) \leq a(1 + b + c + bc) + b(1 + a + c + ac) \\
&\Rightarrow c(1 + a)(1 + b) \leq a(1 + b)(1 + c) + b(1 + a)(1 + c) \\
&\Rightarrow \frac{c}{1+c} \leq \frac{a}{1+a} + \frac{b}{1+b}.
\end{aligned}$$

Let $x, y \in X$. Then

$$\widehat{\rho}(x, y) = \frac{\rho(x, y)}{1 + \rho(x, y)} < \frac{\rho(x, y)}{\rho(x, y)} = 1;$$

thus $H = \sup\{\widehat{(\rho)}(x, y) \mid x, y \in X\} \leq 1$.

Suppose that $G = \infty$, and let $\epsilon > 0$. Then there exist $x, y \in X$ such that $\rho(x, y) > \frac{1}{\epsilon} - 1$. Now

$$\begin{aligned}
\rho(x, y) > \frac{1}{\epsilon} - 1 &\Leftrightarrow 1 + \rho(x, y) > \frac{1}{\epsilon} \\
&\Leftrightarrow \frac{1}{1 + \rho(x, y)} < \epsilon \\
&\Leftrightarrow 1 - \frac{\rho(x, y)}{1 + \rho(x, y)} < \epsilon \\
&\Leftrightarrow 1 - \widehat{\rho}(x, y) < \epsilon.
\end{aligned}$$

Since this is true for every epsilon, Thus $H = \sup\{\widehat{(}\rho(x, y) \mid x, y \in X\} \geq 1$. Combined with part **(b)**, we have $H = 1$.

Suppose that $X$ is finite. Then the set $\{\rho(x, y) \mid x, y \in X\}$ is also finite, and thus has a maximum, and this maximum is equal to $G$. Then there exist $a, b \in X$

such that $\rho(a,b) = G$. Since $f(x) = \frac{x}{1+x}$ is an increasing function, $\rho(a,b) \geq \rho(c,d)$ implies that $\widehat{\rho}(a,b) \geq \widehat{\rho}(c,d)$. Thus

$$\frac{G}{1+G} = \widehat{\rho}(a,b) = \max\{\widehat{\rho}(x,y) \mid x,y \in X\} = H.$$

□

**Exercise VI.8.** Let $(X,\rho)$ be a metric space. Let $x \in X$ and let $A, B \subset X$ be neighborhoods of $x$. Show that $A \cap B$ is a neighborhood of $x$.

*Solution.* Since $A$ and $B$ are neighborhoods of $x$, each contains an open set which contains $x$; say $x \in U \subset A$ and $x \in V \subset B$ with $U$ and $V$ open. Then $U \cap V$ is open, contains $x$, and is a subset of $A \cap B$. Thus $A \cap B$ is a neighborhood. □

**Exercise VI.9.** Let $\mathbb{S}^1$ be the unit circle together with the subspace metric inherited from $\mathbb{R}^2$. Let $(a_n)$ be the sequence in $\mathbb{S}^1$ defined by

$$a_n = \left(\cos\frac{2\pi n}{6}, \sin\frac{2\pi n}{6}\right).$$

Find the cluster points of $(a_n)$.

*Solution.* The sequence $(a_n)$ takes exactly the six values

$$\{(\pm 1, 0), (\pm\frac{1}{2}, \pm\frac{\sqrt{3}}{2})\}.$$

Each of these values occurs infinitely often, so this is the set of cluster points. □

**Exercise VI.10.** Let $X$ be a set and define a metric $\rho$ on $X$ by

$$\rho(x,y) = \begin{cases} 0 & \text{if } x = y; \\ 1 & \text{otherwise .} \end{cases}$$

Let $(a_n)$ be a sequence in $X$.

**(a)** Show that $p \in X$ is a limit point of $(a_n)$ if and only if

$$\exists N \in \mathbb{N} \mid n \geq N \Rightarrow a_n = p.$$

**(b)** Show that $q \in X$ is a cluster point of $(a_n)$ if and only if

$$\forall N \in \mathbb{N} \exists n \geq N \mid a_n = q.$$

*Solution.* In a discrete metric space, the singleton set $\{x\}$ is a neighborhood of $x$. Now $p$ is a limit point if and only if $\exists N \in \mathbb{N} \mid n \geq N$ implies that $a_n$ is in $\{p\}$; this happens exactly when $a_n = p$ for $n \geq N$. Thus **(a)**. Clearly **(b)** is similar. □

**Exercise VI.11.** Find an example of a sequence $(a_n)$ of real numbers and a real number $q \in \mathbb{R}$ such that $(a_n)$ clusters at $q$ but does not converge to $q$.

*Solution.* Let $a_n = (-1)^n$. Then $(a_n)$ clusters at 1. to see this, let $U$ be a neighborhood of 1, and note that for all even $n$, of which there are infinitely many, we have $a_n = 1 \in U$. By **C4**, $(a_n)$ clusters at 1.

However, $(a_n)$ does not converge to 1, because for all odd $n$, of which there are infinitely many, we have $a_n = -1 \notin U$. Since **L4** is not satisfied, $(a_n)$ does not converge to 1. □

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